Fourth International Derive TI-89/92 Conference
Liverpool John Moores University, July 12 – 15, 2000

Construction of Mathematical Concepts and the Use of SymbolicCalculators

Fernando Hitt
Departamento de Matemática Educativa del Cinvestav-IPN, México
Email: fhitta@data.net.mx

Summary

In this paper we intend to do an integration of theoretical aspects involving the construction of mathematical concepts and their implications in the teaching of Mathematics. We expound problems met by students when they make no reflexive use of technology in the solution of problems, and mention the importance of building environments richer than the so called "paper, pencil and technology", that go along with theoretical aspects promoting the learning of Mathematics.

Problems involved in the learning of Mathematics

Research on problems involving the learning of mathematics show complex characteristics arising in the construction of concepts and their application in the solution of problems. In order to illustrate such problems we shall analyse various studies, beginning with a study by Selden et al (1989, 1994) dealing with non routine problems.

Selden et al carried out their first study (1989) with Engineering students who had passed a calculus course with a "C" (the highest grade being "A", then "B", then "C"…). Students were given five non routine problems to solve, one of them being:

Does $x^{21}+x^{19}-x^{-1}+2=0$ *have real roots between –1 and 0? Why or why not?*

Selden et al point out that "*not one of the engineering regular students* [University of Tennessee], *could fully solve any of the non routine problems proposed to them*".

Later in their study of 1994, they considered the same five non routine problems, but this time including in the sample some students that had passed a course on calculus with grades "A" and "B", and the results were not much better than was expected vis-à-vis the 1989 study results. What must have happened with these students? Before inferring anything let us analyse the non-routine problem above mentioned. The equation involved $x^{21}+x^{19}-x^{-1}+2=0$ is not an algebraic expression that may evoke some solving procedure at sight. This is precisely the non-routine aspect which, in the absence of a familiar procedure, leads the student to the use of diverse representations of the mathematical object under study. Selden et al (1994, p. 25) said:

> *Many students in this study preferred arithmetic and even quite sophisticated algebraic techniques for solving calculus problems… Graphs were used in only one-fourth of the solutions attempts to the first four non-routine problems. As others have observed, proper interpretation of graphs appears to be quite difficult.*

One of the procedures the students could have followed was proposing a function, that of $f(x)=x^{21}+x^{19}-x^{-1}+2$, evaluating the function in $f(-1)=1>0$ and showing that the tangent slope at each one of the corresponding points of the curve for $-1\le x<0$ were positive. That is, the derivative of the function was positive in the points of the domain such that $-1\le x<0$ Calculating

$f'(x) = 21x^{20} + 19x^{18} + x^2$, it appears that the function is positive for every point $x \neq 0$, since all of the exponents of the polynomial are pair numbers and the coefficients are positive.

Therefore, $f'(x) > 0$, *para toda* $x \in [-1, 0)$. As a consequence, the function f is strictly an increasing one in the interval $[-1, 0)$ and because it is $f(-1) = 1 > 0$ it has no real root in this interval.

But does the previous algebraic process not imply having a previous idea of the function's behaviour? How can we, in such a case, carry out algebraic processes without geometric ideation support? (i.e. Figure 1).

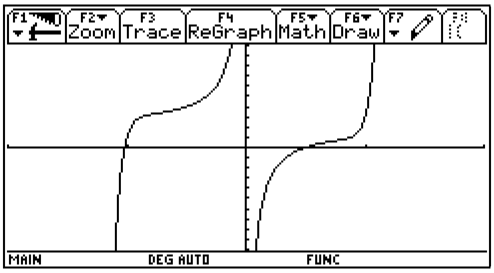


Figure 1

Selden et al point out that students of both studies were trained by experienced teachers who followed an ordinary school text. In general it seems that students approached the non-routine problems confining themselves to algebraic processes fundamentally.

Do Mathematics teachers promote visual considerations in their calculus courses?

If we assume Eisenberg and Dreyfus' perspective as well as Vinner's (1990), among others, we can say that students' avoidance of visual considerations (certainly promoted by teachers' teaching) has held them from exploring graphic possibilities in order to understand, place and possibly find a thinking line that might lead them to the problem solution. We shall revisit this issue later on.

What would have happened if these students had been allowed to use a calculator and had answered that as the equation associated function is graphed, it follows from the screen image that there is no root in $[-1, 0)$?

Let us check another example that illustrates the problem that arises from confining oneself to only one representational system. Let us assume we wish to analyse a polynomial in order to find the points over the domain where the polynomial is nullified.

$$f(x) = -18x^3 + 2x^6 + 2x^4 - 18x^2 + x^5 - x^6 - 20 - 19x - x^6$$

As we resort to the usual algebraic reducing and ordering operations, we may reach an equivalent expression.

$$f(x) = x^5 + 2x^4 - 18x^3 - 18x^2 - 19x - 20$$

We may now assert it is a fifth grade polynomial. As we stay within the algebraic system, we may resort to the rational roots search methods through the use of synthetic division.

As we analyse the coefficients' sign change according to Descartes' Rule, we have a sign change of coefficient +2 de x^4 al –18 de x^3 . Therefore, there must be a positive real root. Let us now calculate $f(-x) =$ in order to find real negative roots. Let us see the sign changes among the coefficients: $f(-x) = -x^5 + 2x^4 + 18x^3 - 18x^2 + 19x - 20$. This indicates that there are four sign changes: –1 de x^5 a +2 de x^4, +18 de x^3 a –18 de x^2, –18 de x^2 a 19 de x, *y* 19 de x a –20. $18x^3 - 18x^2 + 19x - 20$. Therefore, there must be four real negative roots or two or zero. We may check if there is some integer or rational root (p/q) analysing factors of $a_n = 1$ y $a_o = -20$. If there is a rational root p it must be a factor of a_o, and q a factor of a_n. Then p may equal: ±1, ±2, ±4, ±5, ±10, ±20. And q may equal 1. Using the synthetic division method to find values p/q which are: ±1, ±2, ±4, ±5, ±10, ±20, it follows that roots are: –1, +4, –5, i, –i . Therefore, $f(x) = (x+1)(x-4)(x+5)(x^2+1)$.

The following exercise is one where a student making no errors along the algebraic process would possibly have no need to use his graph calculator to reach a solution. We have reduced the algebraic process to a few lines but do you consider students will make not one mistake along the whole algebraic process?

We believe that the graph (See Figure 2) may be of great help. Nevertheless, the process of graphing the polynomial provides relevant information when transposing an algebraic exercise into visual information. What would happen if a student who is graphing arrives at the conclusion that roots are –1, +4 y –5 because "one can see" that the function turns zero at those points? What to do in this case? Is it convenient that students use a graphing calculator for such an exercise? Would a calculus teacher admit such an answer as "partial"? It is here where we find the great controversy.

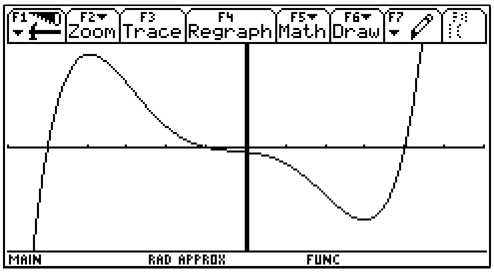


Figure 2

We have presented two situations here: the work of Selden et al that stresses the need to move towards graphic representation to orient steps in problem solving. The second example discusses the problems that arise from reducing an exercise to a an act of perception in such a way that if a student uses the information on the screen in an efficient way, the only task for him to do will then be finding the imaginary roots.

Now we shall advance some theoretical aspects in reference to the role of representation and visualisation in mathematics.

Mathematics representation and visualisation semiotic systems.

The development of constructivism in this computational era has strongly promoted the study of semiotic representations and their relationship with the construction of mathematical knowledge. From this point of view, authors like Skemp (1986), Janvier (1987), Kaput (1991) and Duval (1993, 1995) have thoroughly analysed the role played by mathematical representations in building concepts. At the same time, authors like Eisenberg and Dreyfus (1990), Zimmermann and Cunningham (1990), Vinner (1989), among others, have studied the role of mathematical representation in the frame of mathematical visualisation. In reference to the problems dealt with in this study see Zimmermann (1990, p. 136), he said:

> *Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding, which is widely recognized to be lacking in many calculus courses as now taught. Symbol manipulation has been overemphasized and … in the process the spirit of calculus has been lost.*

All of the above mentioned authors stress the relevance of using varied representation in the construction of mathematical objects as well as in problem-solving. Duval (1993, 1995) asserts: *"…we are then faced with what could be called the cognitive paradox of mathematical thinking. On the one hand, the apprehension of mathematical objects can only be conceptual apprehension, and on the other, only through semiotic representation is it possible to do an activity involving mathematical objects".* Duval (1993, 1995) proposes Figure 3 to stress the importance of tasks involving conversion between representational systems for the construction of concepts.

In reference to Figure 3, we may point out that Duval considers the notion of treatment on a register as basically associated with the transformations of mathematical objects on a representational system which are carried out within the system itself. He also proposes that working on one system only is not enough and emphasizes that apart from treatment tasks on a given register, the construction of a concept must entail conversion tasks between representational systems (both ways).

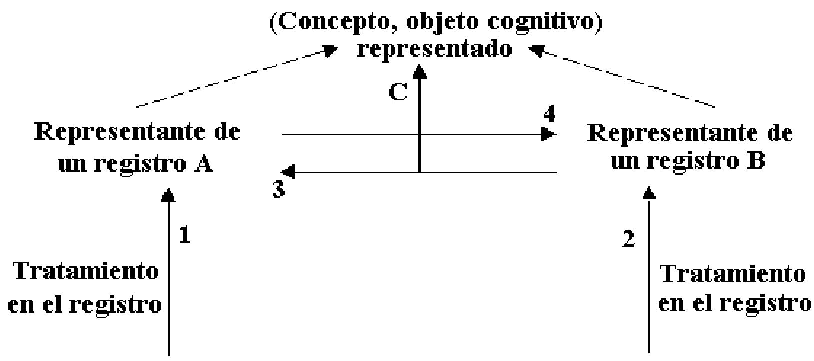


Figure 3

Such theoretical aspects lead us to a relevant problem in the learning of Mathematics. Both empirical research on the problems involved in the learning of Mathematics as well as theory developed thereafter, give high priority to the use of multiple representations of mathematical objects when building concepts and solving problems. **It is worth stressing that conversion between systems of representation is crucial in building concepts.**

Having in mind the previous theoretical support, we must then deal with the issue of making reflexive use of non algebraic representations in "paper, pencil and calculator" environments". In the following example let us consider a graph that may be concealing a fact. For example, given functions $f(x)=3\cdot x^2+2$ and $g(x)=x^3-x^2-6\cdot x$, as we represent them graphically, we obtain what we can see in Figure 4. On the graph, we can see that there is an intersection of the two functions between –0 and 0, when in fact the intersection (See Figure 5) occurs when the value x is 5.22 approximately.

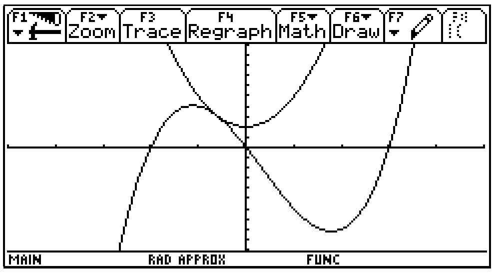


Figure 4

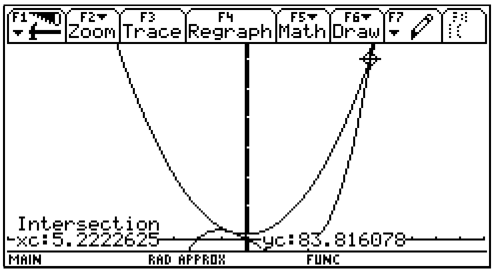


Figure 5

There are many cases dealing with the use of technology in the Mathematics class. For example, Guin and Trouche (1998) set forth the difficulties met by their students on attempts to solve equation $tg(x)=x$, on R. As they assert: *"In a class of 32 students (age 17) only four students arrived at an infinite number of solutions…. The others mentioned a finite number of answers (which are those on the screen)"* (See Figure 6). In reference to equation $\frac{\sin x}{x}=0,\ en\ [0,600]$, see Figure 7, they point out the following: *"Only 10% of 40 terminal and scientific university first year students (age 18 and 19 respectively) answered 'it is nullified every time sin(x) is zero'"*.

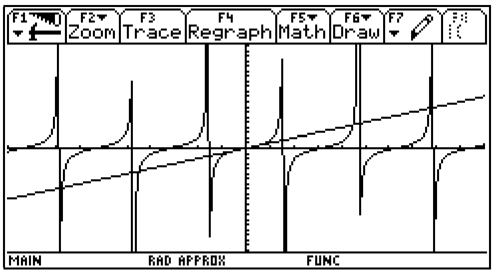


Figure 6

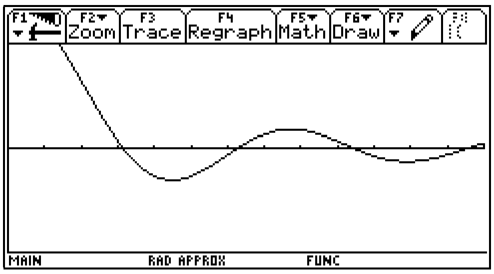


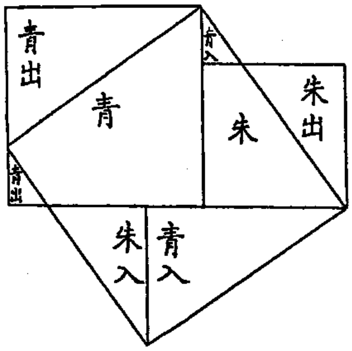
Figure 7

Are graphic representations then not enough to hold a mathematical assertion?

My opinion is that the different representations of a mathematical concept are absolutely necessary. Since geometrical representations are not all the way reliable (See Figure 8), we must complement them with algebraic support. In fact, this is Duval's previously mentioned stance.

Do you agree with the visual demonstration of Pythagoras' theorem?

Visual demonstration of Pythagoras' theorem (Liu Hui, 270 b.C.)



Do you agree with visual demonstration of:
$8 \times 8 = 5 \times 13$?

Visual demonstration of:
$8 \times 8 = 5 \times 13$
Lewis Caroll (1832-1898)

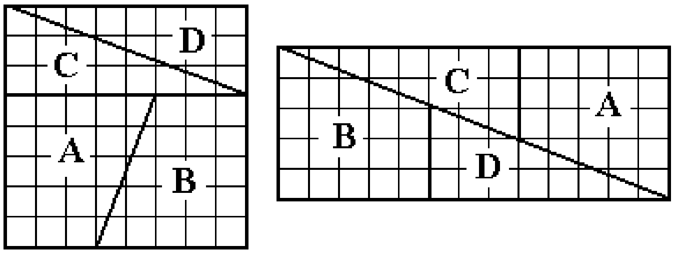


Figure 8

Malabar et al. (1998, p.2) said:

"When given a problem such as $\int_{-\pi}^{\pi} \sin(x)\,dx$*, how many students automatically perform the integration and input the limits rather than 'see' that the answer is zero by consideration of the sine graph?* (idem, p.4), *"Malabar attempted to see if visual skills and, more important, an appropriate balance between symbolic manipulation and visual thinking could be achieved by using visual software and a 'constructivist' approach to the teaching of function and in particular function transformations".*

Also, related to this problem Guin et Trouche (1999) said: *"For example, how to obtain and read a graph (i.e. window manipulations) is missing from mathematics curriculum in France: students must acquire these types of skills by themselves outside of class time".*

Are there any exercises where using technology may not advisable?

To analyse their reactions, a study was carried out where 300 students were presented an exercise that might lead some of them to a contradictory situation. The exercise, was the following:

a) Solve the following inequality: $0.2\,(0.4\,x + 15) - 0.8x \le 0.12$
b) Verify that $x = 10$ is an element of the problem-solution set.

Some students made mistakes on their attempts while others gave correct answers, as follows:

$$0.2\,(0.4x + 15) - 0.8x \le 0.12$$

$0.8x + 0.30 - 0.8x \le 0.12$
$\cancel{0.8x} + 0.30 - \cancel{0.8x} \le 0.12$
$0.30 \le 0.12$

Solution: true for all x No solution

$0.08x + 3 - 0.8x \le 0.12$
$3 - 0.72x \le 0.12$
$-0.72x \le 0.12 - 3$

$x \ge \dfrac{-2.88}{-0.72}$

Solution: $x \ge 4$

$0.08x + 15 - 0.8x \le 0.12$
$15 + 0.72x \le 0.12$
$0.72x \le 0.12 - 15$

$x \le \dfrac{14.88}{0.72}$

Solution: $x \le 20.666...$

Figure 9

Table 1
Answers to the inequality exercise (both items)

	Correct	Incorrect	Total
Results	71	229	300
Number of students who changed strategy upon encountering contradiction.	16 / 71	13 / 229	29 / 300

When using a calculator with symbolic manipulation, the exercise misses the didactic component to which it was designed (See Figure 10). Applying this kind of exercise as team work in a paper and pencil environment is advisable to promote discussion for a correct answer.

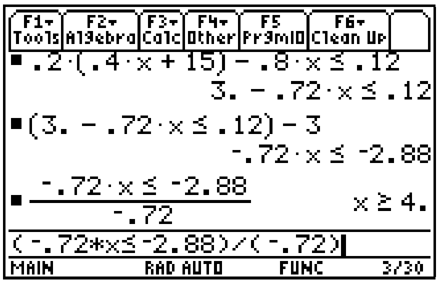


Figure 10

Final reflections

We have here exposed problems involved in non-routine exercises (i. e., Selden et al., 1989, 1994) or entailed by the use of technology. (i.e., Guin and Trouche, 1998, 1999; Malabar et al., 1998). In our experience, the technology-based experiments we have applied to mid-education Mathematics teachers (See Hitt, 1996) have brought to light the pros and cons of the use of technology by teachers themselves. The problems that students encounter are also met by the teachers. As to the above mentioned experiments, we have made the following observations:

Negative aspects	
With no technology	Using technology
• Errors made in algebraic processes are not easily perceived. • Algebraic processes do not promote interaction with geometrical representations.	• Promotes searching for an answer through the trial and error method. • Inhibits analytical thinking in the face of graphic representation. • In some cases, it trivialises the problem turning it into a routine exercise.

Positive aspects	
With no technology	Using technology
• Analytic and algorithmic thinking are promoted in solving problems from an algebraic point of view.	• Allows for the geometrical construction of a given situation. • Allows for visualisation of results produced in an algebraic process. • Some times it is possible for the student to visualise his error. • Graphic representation is possible which helps predicting results. • Symbolic manipulation is possible which allows for concentration on more complex tasks, thus promoting further conceptual learning. • Increases interest in learning Mathematics.

This leads to the importance of carrying out profound studies on the construction of mathematical knowledge and the use of technology, and on the search for “paper, pencil and calculator” activities enhancing the positive aspects in the process of learning Mathematics as mentioned in the table. Also, theory underlying such learning (See Duval, 1993, 1995) should promote articulation among diverse representations of the mathematical objects under study.

References

Duval Raymund (1993) Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Science Cognitives* 5(1993) 37-65.

Duval R. (1995) *Sémiosis et pensée humaine: Registres sémiotiques et apprentissage intellectuels.* Peter Lang, Suisse.

Eisenberg T. & Dreyfus T. (1990) On the Reluctance to Visualize in Mathematics. *In Visualization in Teaching and Mathematics* (Zimmermann W. & Cunningham S. Editors), MAA Series. USA.

Guin D. & Trouche L. (1998) Environnements "Calculatrice symbolique": Necessité d'une socialisation des processus d'instrumentation. Evolution des comportements d'eleves au cours de ces processus. *Actes du Colloque Francophone Européen Calculatrices symboliques et* *géometriques dans l'enseignement des mathématics* (Dominique Guin Ed.). IREM de Monntpellier, France.

Guin D. & Trouche L. (1999) The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning **3**, pp. 195-227.*

Hitt F. (1978) *Comportement de "Return in arrière après la decouverte d'une contradiction"*. PhD. Dissertation, Université Louis Pasteur, France.

Hitt F. (1996) Résolution des problèmes non-routiniers et utilisation des nouvelles technologies. *Proceedings of the International Commission for the Study and Improvement of Mathematics Teaching* (C. Keitel, U. Gellert, E. Jablonka & M. Müller, eds.), Berlin.

Janvier C. (1987) Representation and Understanding: The notion of Function as an example. C. Janvier (ed.), *Problems of Representation in the Teaching and Learning of Mathematics*, Lawrence Erlbaum Associates., p. 67-71.

Kaput J. (1991) Notations and representations as Mediators of Constructive Processes. In Von Glasersfeld E. (ed.*) Radical Constructivism in Mathematics Education*, Kluwer Academic Publishers, pp. 53-74.

Malabar I., Pountney D. C. & Townend M. S. (1998) Combining Visual and Symbolic Skills in the Teaching and Learning of Mathematics. *Proceedings of the 3rd International Derive and TI-92 Conference*, Gettysburg, Penn.

Selden J., Mason A. & Selden A. (1989) Can Average Calculus Students Solve Nonrutinier Problems?. Journal of Mathematical Behavior, 8 (1989) 45-50. Traducción en Antología en Educación Matemática (compiladores R. Cambray, E. Sánchez y G. Zubieta). DME-Cinvestav, 1992, México, p. 77-83.

Selden J., Mason A. & Selden A. (1994) Even good calculus students can't solve nonroutine problems?. In *Research Issues in Undergraduate Mathematics Learning* (J. Kaput & De Dubinsky), MAA Notes Number 33, 1994, p. 19-26.

Skemp R. (1986) *The Psychology of Learning Mathematics*. Penguin Books, Second Edition.

Vinner S. (1989) The Avoidance of Visual Considerations in Calculus Students. Focus on Learning Problems in Mathematics, Vol. 11, pp. 149-156. Traducción en *Antología en Educación Matemática*, compiladores : Cambray R., Sánchez E. y Zubieta G., Dpto. Matemática Educativa, Cinvestav-IPN, México.

Zimmermann W. & Cunningham S. (1991) What is Mathematical Visualization?. *In Visualization in Teaching and Learning Mathematics* (Zimmermann W. & Cunningham S. Editors), MAA, No. 19.

Zimmermann W. (1991) Visual Thinking in Calculus. *In Visualization in Teaching and Learning Mathematics* (Zimmermann W. & Cunningham S. Editors), MAA, No. 19.