Fourth International Derive TI-89/92 Conference
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When the TI89 or the TI92 helps solving Problems Of the Week (POW) …

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Introduction.

I am teaching mathematics to students whose ages range from 15 to 17, in a french speaking secondary school in Brussels.

In October 1999, my school was equipped with an access to Internet. Since then, whenever I find an interesting mathematical site on the Web having some connection with the curriculum, I encourage my students to visit it.Among those interesting sites, there is one I recommended many times this academic year :

"Problems Of the Week" or http://forum.swarthmore.edu/pow/.
The initial aim was to help the students to improve themselves in problem solving by stimulating them to try to solve at least one of the problems every week.

In the beginning, the enterprise had only a limited success. Only a few students were enthusiastic about it, the majority of them did not even collect the problems, sometimes because they were in English !

In order to persuade the ones who were reluctant that they were missing an occasion to be smarter in mathematics and in order to maintain the enthusiasm of the others, I sometimes took a "past problem of the week" as subject of a lesson. I soon realized that the POW's represent for the teacher an enormous collection of material to work on pedagogically.

In addition, some of the POW's can be studied using calculators to help investigations or simply to make calculations.

In the following pages you will find three problems that were treated in the classroom using TI 92 or TI 89 calculators. We will focus on different aspects :

- how to benefit in the class from solutions already found by students;
- how to go a little further than requested in the problem;
- how to take advantage of the technology to "push" some weaker students towards a solution or to generate new questions.

Trigonometry and Calculus Problem of the Week.

Making Money at the Movies - posted April 17, 2000

You are the business manager at a local movie theater. Based on past ticket sales, you know that the theater averages 5,000 customers per day. The current ticket price for all show times is $5 per person. There are no discounted prices for children, students or senior citizens.

The general manager of the theater wants to increase prices in order to increase profits. However, he is afraid that if the price is increased too much, you could end up loosing money. He has asked you to determine how much you should raise the price in order to maximize your profit. Your current profit is $2 per ticket. Any increase in ticket price would be a direct increase in profit.

You randomly survey customers over a period of 2 weeks. Your results show that you would lose 800 customers per every 50 cent increase in ticket price. For example, if you increase the ticket price to $6, you would lose 1600 customers.

What new ticket price would lead to a maximum profit? Be sure to include a complete and detailed explanation of what you did to solve this problem.

This problem is either a good illustration of regression curves or a nice exercise on the chapter about the function $f(x) = ax^2 + bx + c$.

1. Application of regression curves, using the TI 89.

Let us create a Data table containing the following columns :

1^{st} column : price of the ticket
2^{nd} column : number of customers
3^{rd} column : total benefit

We either fill in every column, line by line (see figure 1) …

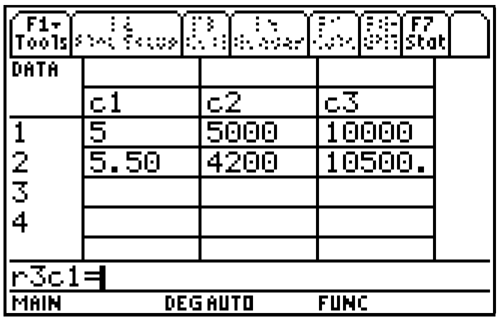

Figure 1

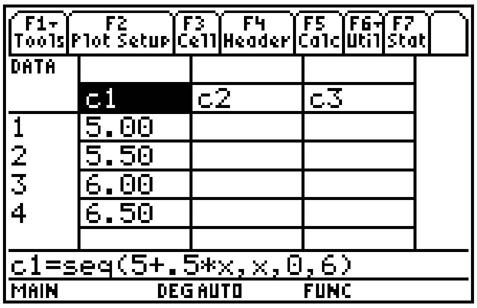

Figure 2

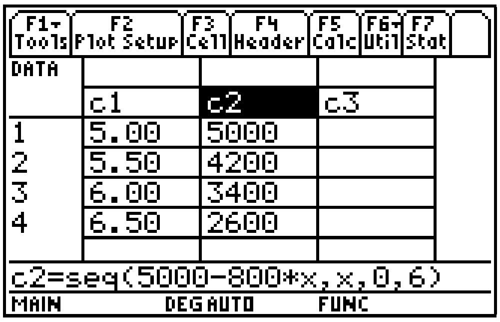

Figure3

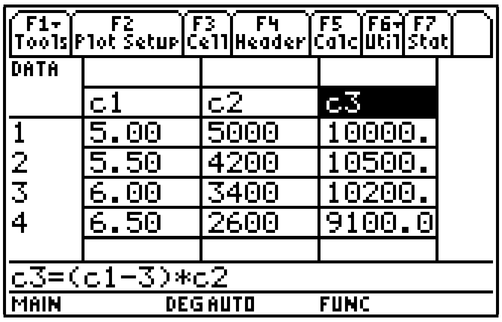

Figure 4

or we do it in a more elegant manner (see figures 2, 3 and 4) :

Why does x vary between 0 and 6 ?

The price of the ticket has to remain positive !

And so, x must be the greatest integer such that

$$5000 - 800x > 0 \Leftrightarrow x < \frac{25}{4} = 6.25$$

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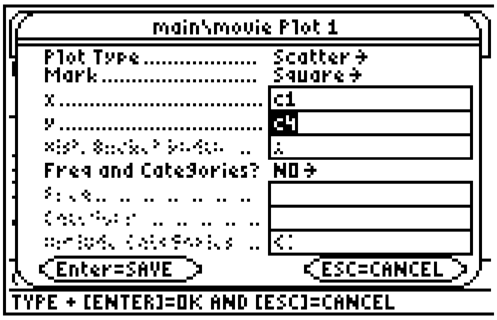


Figure 6

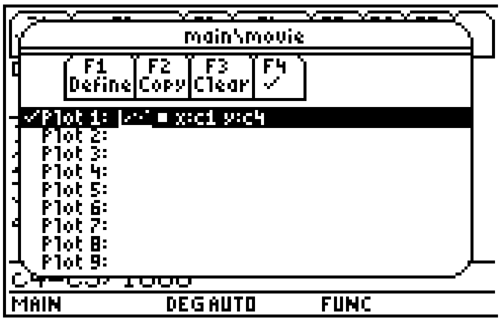
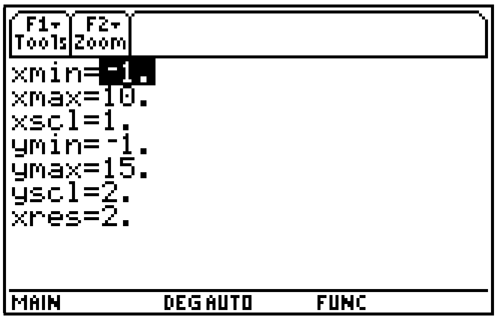
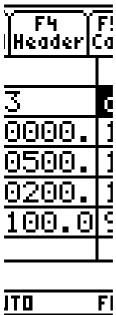


Figure 7

The following step is to sketch the graph of the benefit (column c3) as a function of the price of a ticket (column c1). The numbers representing the benefit are big, so we rather consider the benefit divided by 1000 (see column c4 of figure 5) :





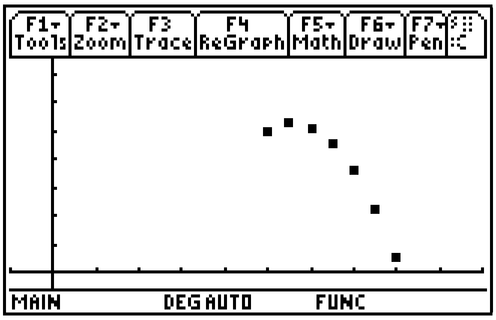


Figure 8

Figure 9

Let us define the type of plot we want to make :

Prepare an adequate Window ….. and sketch the graph of the benefit versus the ticket price

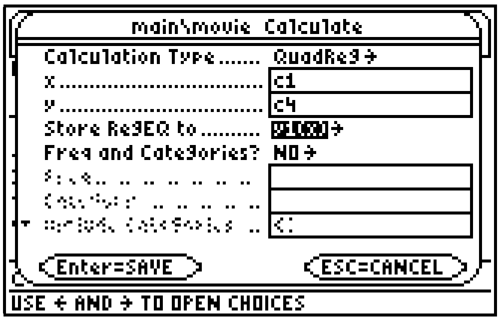


Figure 12

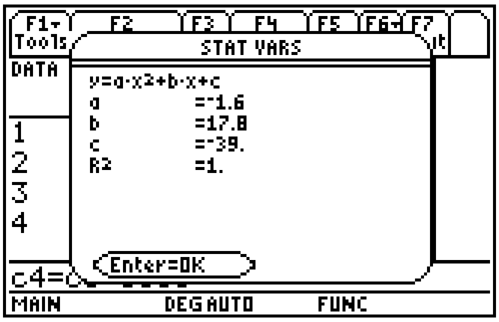


Figure 13

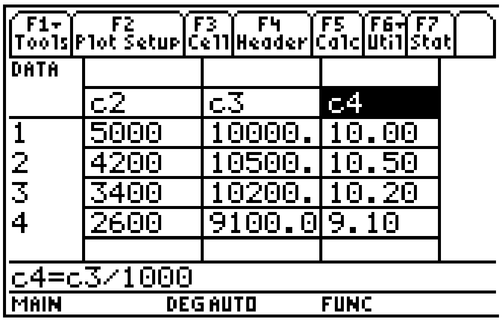


Figure 10

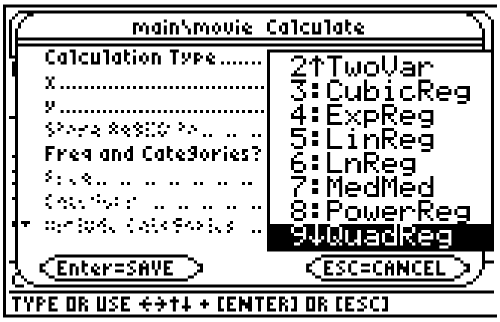


Figure 11

As the graph looks like a parabola, a quadratic regression should be fine !
We go back to the Data table "movie" and choose F5, QuadReg

We ask to store the regression equation in y1(x) and obtain this regression equation giving the graph

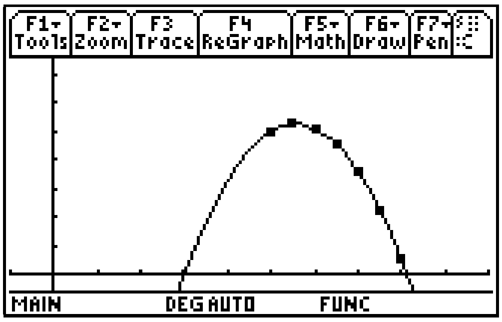


Figure 14

To find the maximum of y1(x),
we calculate the zero of its derivative :

it appears that the optimum ticket price
will be $5.56.

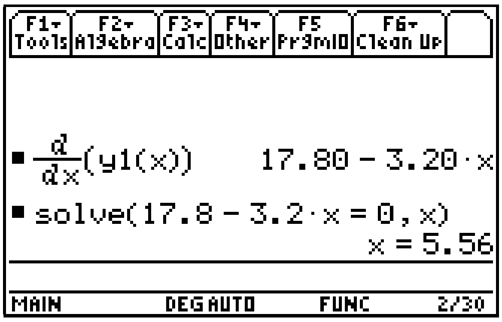


Figure 15

Note that the regression coefficient in figure 13 is equal to 1. This means that the graph of y1(x) goes exactly through all the points or that the benefit versus price x is exactly equal to $1000\cdot(-1.6x^2+17.8x-39)=-1600x^2+17800x-39000$. We would like to have a proof of this.

2. Exercise on the chapter about the function $f(x)=ax^2+bx+c$**.**

Let f(x) be the benefit corresponding to a ticket price x. Give an expression of f(x).

If $x=5+n\cdot 0.5$, then the number of customers will be $5000-n\cdot 800$.

As the benefit per customer is $x-3$, the benefit f(x) will be equal to $(x-3)\cdot(5000-n\cdot 800)$.

But $x=5+n\cdot 0.5 \Leftrightarrow n=\dfrac{x-5}{0.5}=2(x-5)$.

Hence $f(x)=(x-3)(5000-1600(x-5))=-1600x^2+17800-39000$.

If we factorize f(x) we obtain $-200(x-3)(8x-65)$.

The zeros of this polynomial function of degree 2, are respectively 3 and $\dfrac{65}{8}$.

The maximum occurs for $x=\dfrac{3+\dfrac{65}{8}}{2}=\dfrac{89}{16}=5.56$ which corresponds to the answer

of figure 15.

Geometry Problem of the Week

The Return of the Parallelogram - posted January 24, 2000

This is a followup to the problem from two weeks ago - the parallelogram split by the line. Read it carefully, because it's a little bit different.

Draw a parallelogram ABCD with AB=5. Draw EF with E between A and B and F between C and D such that EF divides the area of ABCD in half. If EF=2, what is the area of the parallelogram?

(Here's a hint: the answer doesn't have to be a single number, but tell me as much as you can about the area of the parallelogram. What can it be? I want to know "the answer", which means everything, not just "an answer".)

1. How to place points E and F in order that the area of the parallelogram ABCD would be divided in half ?

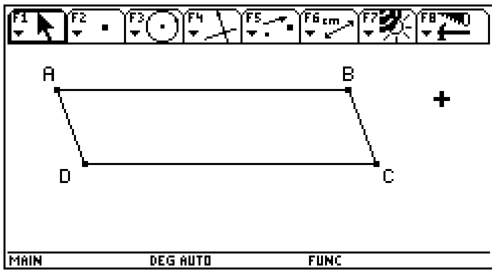


Figure 1

Here is a way to help a weak student unable to solve that problem :

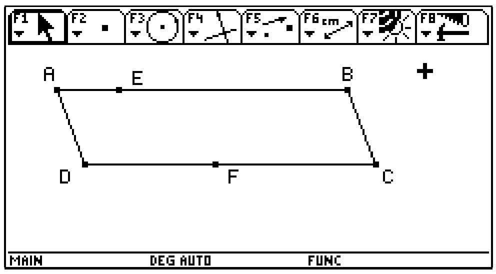


Figure 2

Give him any parallelogram in the Geometry Application of the TI 92;

Tell him

- to draw point E anywhere on side [AB] and F anywhere on side [CD];

- to define 2 polygons AEFD and EBCF;

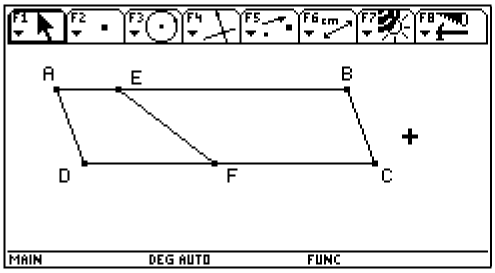


Figure 3

- to ask for the areas of these 2 polygons;

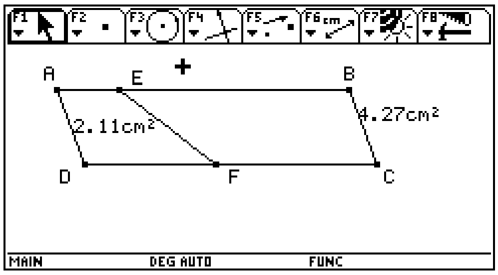


Figure 4

- to move one of the points E or F, observe what happens to the areas and stop when the areas are equal.

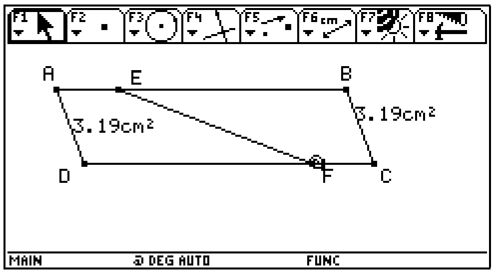


Figure 5

It appears clearly that ABCD is cut in half if and only if $|AE| = |FC|$ i. e. if and only if E and F are symmetric with respect to the center O of ABCD.

It is now the task of the student to prove it !

2. Let us admit that E and F should be symmetric with respect to O and take into account that |EF| must be equal to 2.

Here is a pupil's (Muriel) solution :

- she constructs a parallelogram ABCD with |AB| equal to 5 and with variable height (the area of a parallelogram depending only on its base and height);

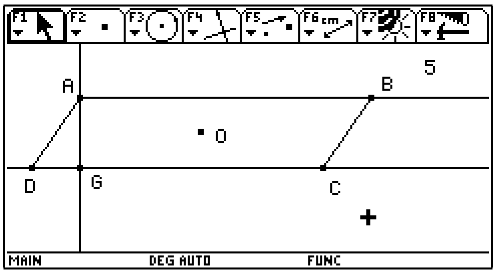


Figure 6

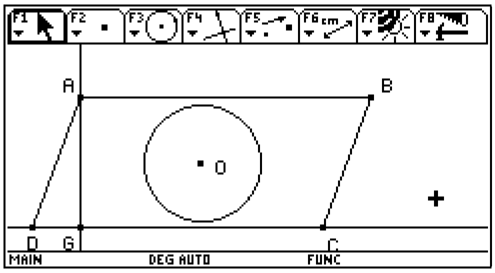


Figure 11

- then, as E and F are symmetric with respect to O and such that |EF| = 2, E and F belong to a circle with center O, radius 1;

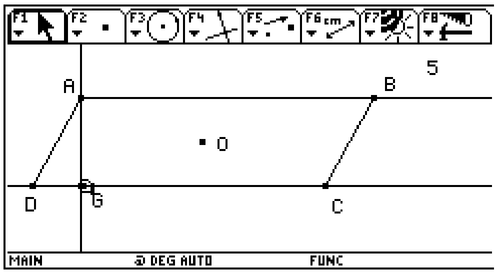


Figure 7

- and hence for the following parallelograms, there are two possible choices for E and F (see figures 9 and 10),

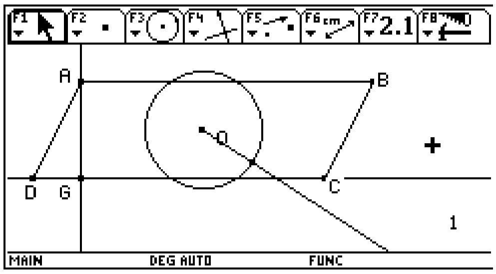


Figure 8

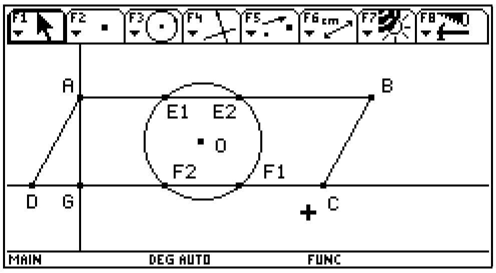


Figure 9

for the next parallelogram (see figure 11), there is no point E satisfying the conditions because its height is greater than 2 !
(should its height be equal to 2, we would find <u>one</u> point E).

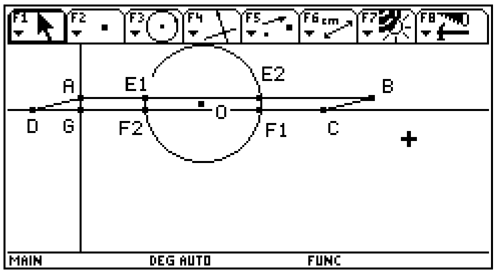


Figure 10

- and so, as the height of ABCD is allowed to vary between 0 and 2, the area of the parallelogram varies between 0 and 20.

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3. Employing Muriel's solution in the classroom.

3.1. *Let us call h, the height of parallelogram ABCD. Muriel has clearly shown that*

$$h > 2 \Rightarrow \text{there are no points E and F such that } \begin{cases} E \in [AB] \text{ and } F \in [DC] \\ \text{points E and F divide the area of ABCD in two equal parts} \\ |EF| = 2 \end{cases}$$

Does this mean that for any given parallelogram ABCD whose height h is smaller than 2, points E and F always exist ?

No ! In the example of figure 12, there are 2 points E satisfying the conditions, in figure 13, there is no point E verifying the conditions.

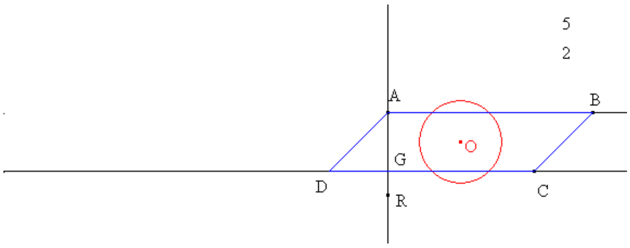


Figure 12

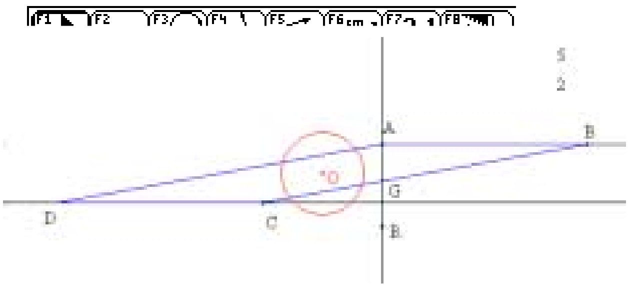


Figure 13

In the two figures, $|AB| = 5$, $|AR| = 2$ and the heights $h = |AG|$ are the same and less than 2.

But $b = |GD|$ is not the same in the two figures.

A new question arises : *given a parallelogram with fixed height $h \le 2$, determine the values of b for which points E and F exist.*

I leave this point to the reader.

3.2. *Choose h and b so that there are two possible positions for E (and hence for F).*

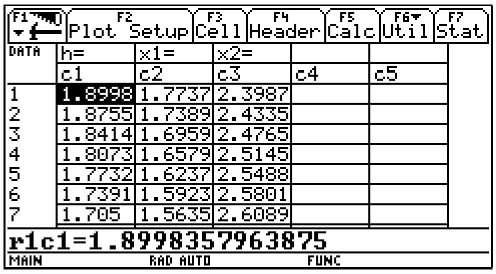
Let x1 be the distance from A to E1, x2 the distance from A to E2.

3.2.1.Collect the values of x1 and x2 in function of h and sketch the graphs of x1(h) and x2(h).

In figure 14, $b = |GD|$ is fixed, h varies

between 0 and 2

- measure h, |AE1| and |AE2| and collect these data in table sysdata;



DATA	h=	x1=	x2=		
	c1	c2	c3	c4	c5
1	1.8998	1.7737	2.3987		
2	1.8755	1.7389	2.4335		
3	1.8414	1.6959	2.4765		
4	1.8073	1.6579	2.5145		
5	1.7732	1.6237	2.5488		
6	1.7391	1.5923	2.5801		
7	1.705	1.5635	2.6089		

r1c1=1.8998357963875

Figure 18

- animate point G (figures 15 to 17);

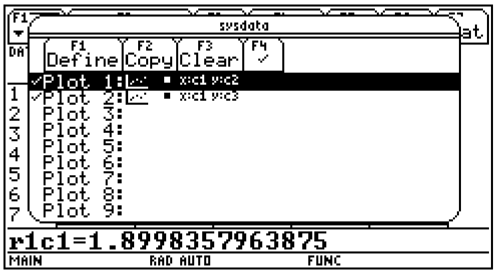


Figure 19

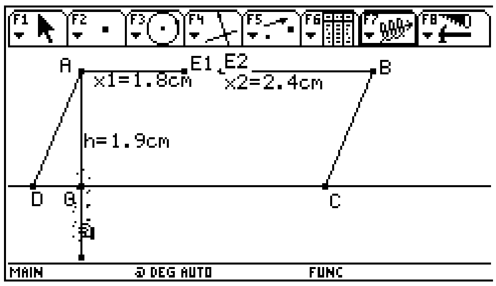


Figure 15

- view table Sysdata;

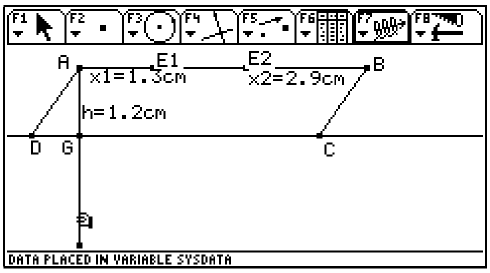


Figure 16

Plot1 : x1 as a function of h,

Plot2 : x2 as a function of h;

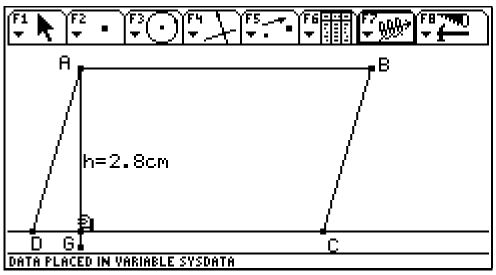


Figure 17

- choose an adequate window;

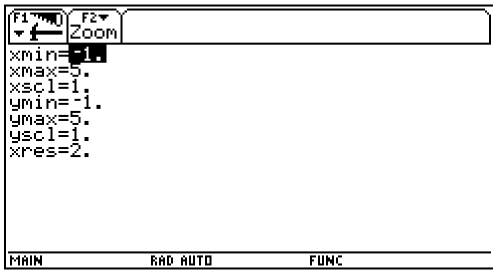


Figure 20

- sketch the graphs;

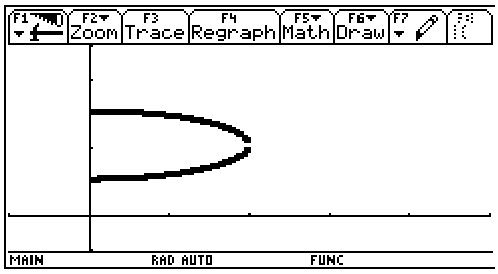


Figure 21

3.2.2. Determine the analytical expressions of x1(h) and x2(h).

(though it was done in the classroom, I leave this point to the reader).

4. Suggesting another construction to the class, to solve the same problem.

Construct a segment [AB] such that |AB| = 5, choose a point E anywhere on [AB].

Construct a parallelogram ABCD such that /EF/ = 2, knowing that F lies on [CD] and is the symmetric of E with respect to the center of the parallelogram.

- construct a segment [AB] of length 5;

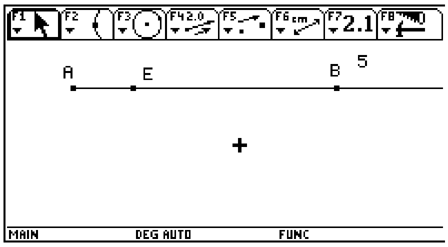


Figure 22

- construct a circle C1 with center E, radius 2; this circle is the locus of F : F could be any point of C1.

  But once F is chosen, where is C ?

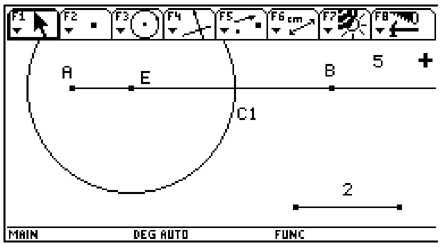


Figure 23

- As $\vec{FC} = \vec{AE}$, C belongs to another circle : the image C2 of C1 by the translation whose vector is $\vec{AE}$;

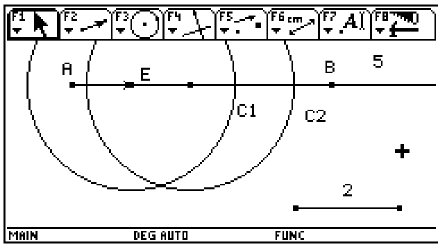


Figure 24

- finally point D belongs to the circle C3, translated from C2 by translation $t_{\vec{BA}}$;

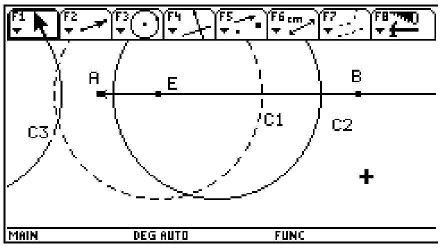


Figure 25

- here is one of the parallelograms that satisfy the conditions. It is clear again that the height of such a parallelogram is maximum 2 which happens when C is situated on the lowest point of circle C2.

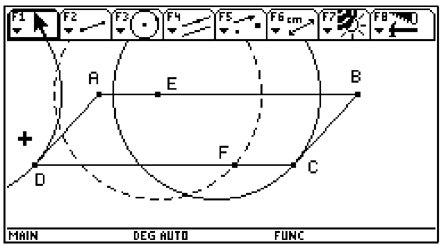


Figure 26

Trigonometry and Calculus Problem of the Week.

Stonybrooke River Bridge - posted March 27, 2000

The Stonybrooke Recreation Department has one special project each year. Last year, they resurfaced the tennis courts and the year before they put lights in at the basketball court. This year it is proposed that a foot bridge be constructed across Stonybrooke River. In order to get cost estimates, you need to know the distance across the river. Since you are a recent college graduate with a minor in mathematics, you have been asked to figure out the distance across the river.

You go to the river with a friend and take some measurements. You and your friend stand 10 feet apart facing each other on the bank of the river. Each of you then turns toward the river until you have direct line of sight to the one large oak tree across the river. You determine that you turned 85 degrees counterclockwise to see the tree. Your friend determines he turned 80 degrees clockwise to see the tree.
Note: If you were standing on the opposite bank, the tree would be between you and your friend.

Based on these measurements, what is the distance across the river? Be sure to give a complete explanation of your solution.

formulas.

Lets make a simple sketch of the situation ….

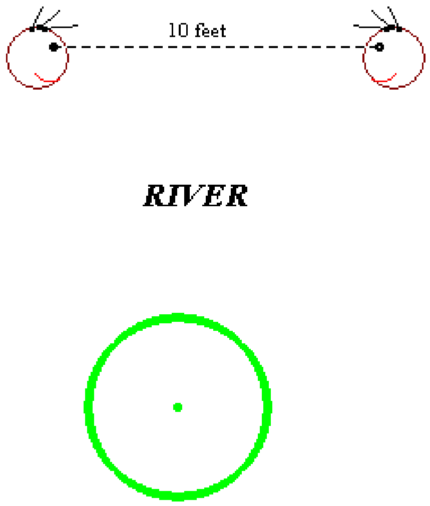


Figure 1

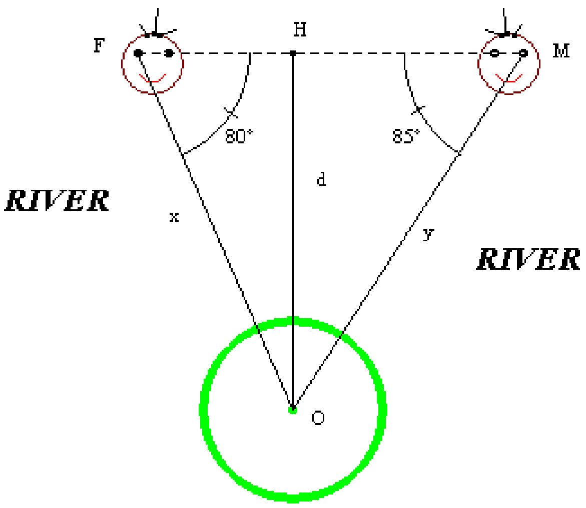


Figure 2

The question is to calculate distance d, knowing that angle F measures 80° and angle M, 85°.

As the angle $F\hat{O}H$ measures 10°, one quickly comes to the conclusion that $d = x \cdot \cos 10°$

with x satisfying the system $\begin{cases} \dfrac{\sin 85°}{x} = \dfrac{\sin 80°}{y} \\ y\cos 85° + x\cos 80° = 10 \end{cases} \Leftrightarrow \begin{cases} y\sin 85° - x\sin 80° = 0 \\ y\cos 85° + x\cos 80° = 10 \end{cases}$ (1)

Solution of Laurent who is not really a fan of calculators, but calculates very well by hand.

He solves system (1) this way :

$$\begin{cases} x = \dfrac{\begin{vmatrix} \sin 85° & 0 \\ \cos 85° & 10 \end{vmatrix}}{\begin{vmatrix} \sin 85° & -\sin 80° \\ \cos 85° & \cos 80° \end{vmatrix}} = \dfrac{10\sin 85°}{\sin 85°\cos 80° + \cos 85°\sin 80°} = \dfrac{10\sin 85°}{\sin 165°} = \dfrac{10\sin 85°}{\sin 15°} \\ y = \ldots\ldots\ldots \end{cases}$$

and comes to the conclusion that $d = \dfrac{10\sin 85°\cos 10°}{\sin 15°}$ (2)

Solution of Nicolas who is a fan of calculators.

- He takes his TI 89, enters system (1) and solves it;

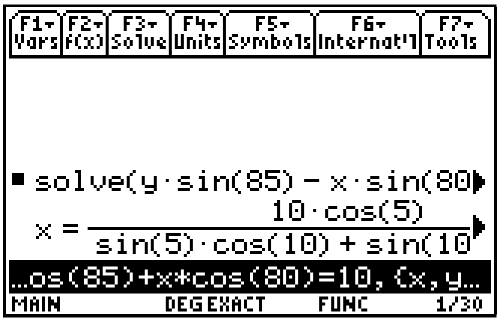


Figure 3

- he then extracts the value of x and multiplies it by $\cos 10°$;

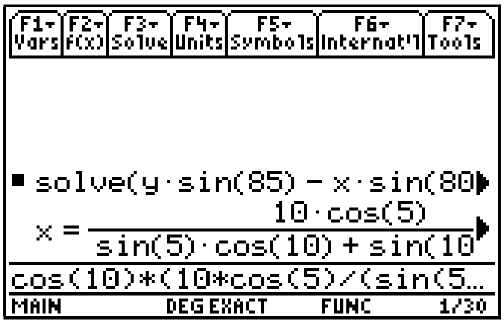


Figure 4

- arriving to the conclusion that

$$d = \frac{10\cos 5°\cos 10°}{\sin 5°\cos 10° + \sin 10°\cos 5°} \quad (3).$$

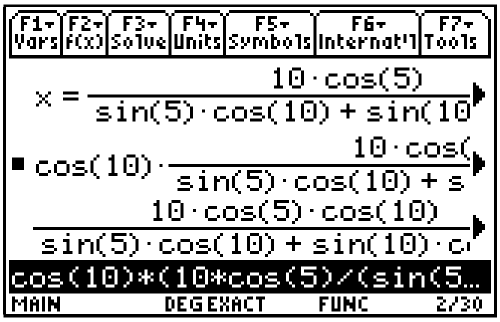


Figure 5

Remark : Nicolas is working in Exact Mode.

Solution of Pierre who knows his TI 89 calculator very well .

- He solves the same system as Nicolas (see figure 3), extracts the value of x and multiplies it by $\cos 10°$;

  (he also works in Exact Mode)

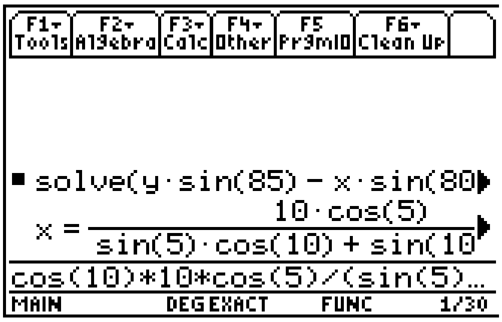


Figure 6

- he chooses F2, Trig ;

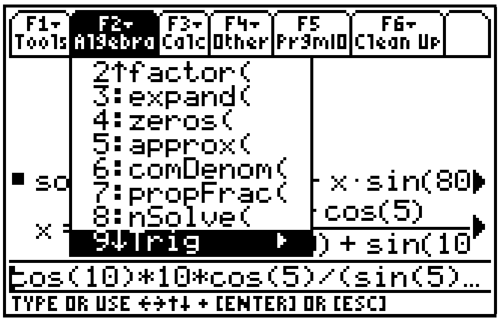


Figure 7

- then tCollect(;

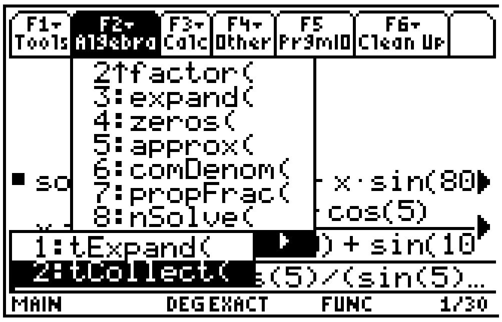


Figure 8

- adds a parenthesis at the end of the expression;

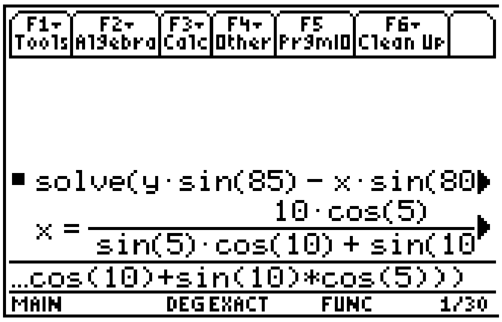


Figure 9

Contents

- finally comes to the conclusion that

  $d = 5[\cos 5°(\sqrt{3}+1)\sqrt{2}+\sqrt{3}+2]$ (4).

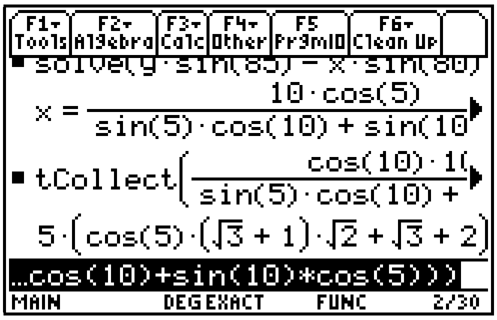


Figure 10

And now the teacher… with her questions !

1. *Prove that answers (2), (3) and (4) are equal.*

Act 1. The first reaction of the pupils is to ask for an approximate value of the three expressions, but this does not mean anything !

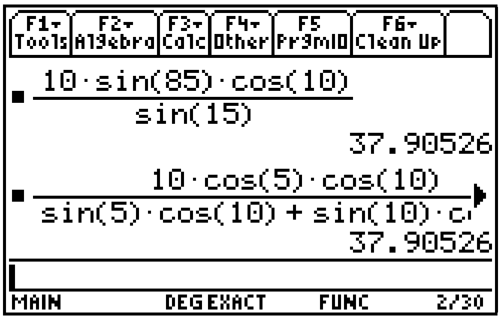


Figure 11

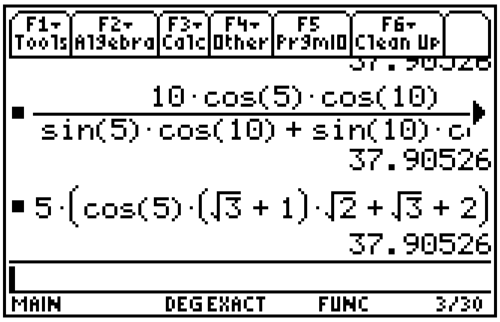


Figure 12

Ir in A]
d

Act 2. It is very easy to prove that (2) is equal to (3) :

$\sin 5°\cos 10° + \sin 10°\cos 5° = \sin 15°$ and $\sin 85° = \cos 5°$ because the angles are complementary.

Act 3. So, we may write that $d = \dfrac{10\cos 5°\cos 10°}{\sin 15°}$ (5)

And we still have to prove that this is equal to $d = 5[\cos 5°(\sqrt{3}+1)\sqrt{2}+\sqrt{3}+2]$ (4)

Contents

Laurent takes his calculator and looks how the machine manages expression (5) :

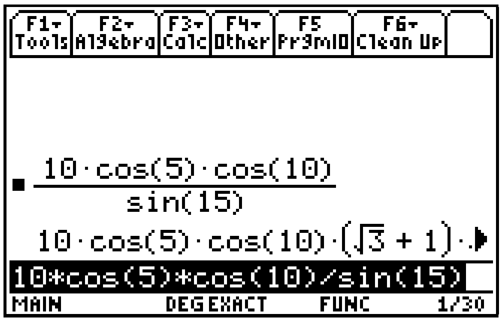
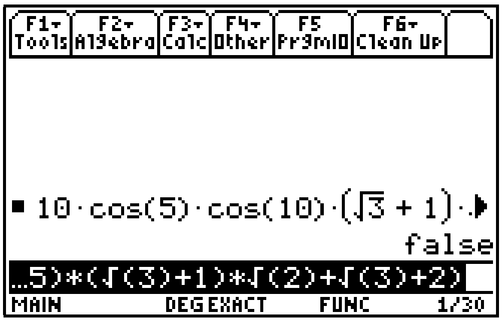


Figure 13

Act 4. Is $10\cos 5°\cos 10°(\sqrt{3}+1)\sqrt{2} = 5[\cos 5°(\sqrt{3}+1)\sqrt{2}+\sqrt{3}+2]$? (6)

Pierre is getting a bit nervous about this, takes the calculator which is connected to the viewscreen so that everybody can follow…



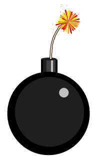


Figure 14

Act 5. The teacher has to give a clue, because the troups are getting discouraged …

The left member of (6) may be written $10(\sqrt{3}+1)\sqrt{2}\cdot\cos 5°\ \cos 10°$

The right member may be written $[5\cos 5°(\sqrt{3}+1)\sqrt{2}]+[5(\sqrt{3}+2)]$

Do you know formulas transforming a product of cosines into a sum ?
Yes, if you read Simpson formulas from right to left…

Act 6. <u>Proof that</u> $10\cos 5°\cos 10°(\sqrt{3}+1)\sqrt{2} = 5[\cos 5°(\sqrt{3}+1)\sqrt{2}+\sqrt{3}+2]$:

According to the formula $\cos p\cdot\cos q = \dfrac{\cos(p-q)+\cos(p+q)}{2}$, we have

$$\cos 5°\cdot\cos 10° = \frac{\cos 5°+\cos 15°}{2}$$

As $\cos 15° = \dfrac{(\sqrt{3}+1)\sqrt{2}}{4}$ (see fig. 15)

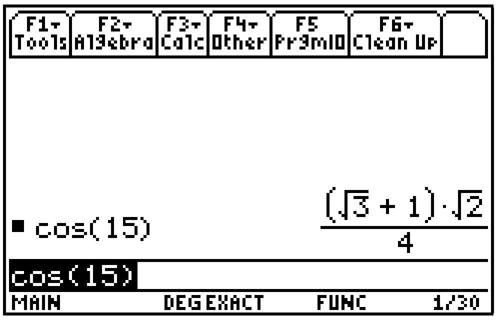


Figure 15

$$\cos 5° \cdot \cos 10° = \frac{\cos 5° + \dfrac{(\sqrt{3}+1)\sqrt{2}}{4}}{2} = \frac{4\cos 5° + (\sqrt{3}+1)\sqrt{2}}{8}$$

and finally

$$10\cos 5° \cos 10° (\sqrt{3}+1)\sqrt{2} = 5 \cdot \frac{4\cos 5° + (\sqrt{3}+1)\sqrt{2}}{4} \cdot (\sqrt{3}+1)\sqrt{2}$$

$$= 5 \cdot (\cos 5° (\sqrt{3}+1)\sqrt{2} + \frac{[(\sqrt{3}+1)\sqrt{2}]^2}{4}) = 5 \cdot (\cos 5° (\sqrt{3}+1)\sqrt{2} + \sqrt{3} + 2)$$

!!!

2. *Why does the calculator know the numerical value of* $\cos 15°$ *and does not know the numerical value of* $\cos 5°$ *?*

Probably because $\cos 15°$ can be obtained as the solution of a simple quadratic equation :

$$2\cos^2 15° - 1 = \cos 30° = \frac{\sqrt{3}}{2}.$$

The actors of the "play"

and the "first roles" :

Here normally comes a photograph!