Contents

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Mechanics of Rigid Body Motions with DERIVE

Leon Magiera
Wroclaw University of Technology, Poland
Email: magiera@rainbow.if.pwr.wroc.pl

Introduction

The majority of the works devoted to the application of the computer algebra systems use as illustrations purely mathematical abstract problems. Only a few of them explore physics problems. It is obvious that performing necessary symbolic derivations with the use of computer algebra system (CAS) significantly reduces the working time and allows a student or teacher to concentrate on physical ideas (the most important part), rather than on the very time consuming technical side, i.e. performing the derivations by hand.

I would like to present the use of DERIVE on the example of mechanics of a rigid body. In particular, I will demonstrate: methods to evaluate the inertia tensor of a rigid body, how to check the formula for the inertial moment of the body, the angular momentum, and the kinetic energy of rotational motion. All of these are expressed by inertia tensor. Also, the calculation methods for the eigenvalue problem (finding the principal axes of the body) are presented.

A rigid body may be regarded as a system of particles in which the distance between any two particles is constant. A particular part in the study of the motion of a rigid body is played by the inertia tensor. This tensor is defined as follows

$$\hat{J} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Its elements are:

$$I_{xx} = \int_V \rho(\vec{r})(\vec{r}^2 - x^2)dV = \int_V \rho(\vec{r})(x^2 + y^2 + z^2 - x^2)dV = \int_V \rho(\vec{r})(y^2 + z^2)dV,$$

$$I_{xy} = I_{yx} = -\int_V \rho(\vec{r})xydV, \text{ and so on,}$$

$\rho(\vec{r})$ is the mass density of the body, $\vec{r}$ is the position vector of the mass element dm and V is the volume of the body.

For discrete mass distribution the integration is replaced by summation:

$$I_{xx} = \sum_i m_i(r^2{}_i - x_i^2) = \sum_i m_i(y_i^2 + z_i^2), \qquad I_{xy} = -\sum_i m_i x_i y_i, \quad \ldots$$

Important relations

For any point mass, m_i, (Fig.1.) located at the position $\vec{r}_i(x_i, y_i, z_i)$ the following quantities are defined:

a) $m_i R_i^2$ - the inertia moment, where the distance $R_i = |\vec{r}_i|\sin(\vec{r}_i, \vec{n}) = |\vec{r}_i \times \vec{n}|$,

b) $\vec{L}_i = m_i(\vec{r}_i \times \vec{v}_i) = m_i(\vec{r}_i \times (\vec{\omega} \times \vec{r}_{i)})$ - the angular momentum,

c) $E_{ki} = \frac{1}{2} m_i v_i^2$ - the kinetic energy.

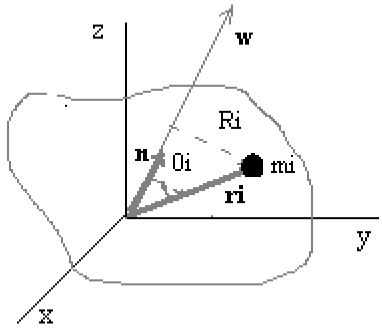

Fig. 1.

DERIVE will be used to prove the following relations:

a)

$$I = \vec{n}^T \hat{I} \vec{n},$$

where I is the inertia moment of a rigid body about the axis defined by the unit vector $\vec{n}$

b) $\vec{L} = \hat{I}\vec{\omega}$, ($\vec{L}$ is the angular momentum, and $\vec{\omega}$ is the angular velocity)

c)

$$E_k = \frac{1}{2}\vec{\omega}^T \hat{I} \vec{\omega}$$

(rotational kinetic energy of the body , "T" denotes matrix transposition).

The contribution of the point mass, m_i, to the inertia tensor is:

$$\hat{J} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where:

$$I_{xxi} = m_i({r^2}_i - x_i^2) = m_i(y_i^2 + z_i^2), \qquad I_{xyi} = -m_i x_i y_i, \quad \text{and so on.}$$

In the preliminary step we enter the above definitions (since the variable name $\hat{I}_i$ can not be used in the DERIVE we replace it by $Ii\_\_$)

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```
[Ixxi:=mi·(yi^2+zi^2),Iyyi:=mi·(Xi^2+zi^2),Izzi:=mi·(Xi^2+yi^2)]
[Ixyi:=-mi·Xi·yi,Ixzi:=-mi·Xi·zi,Iyzi:=-mi·yi·zi]
[Iyxi:=Ixyi,Izxi:=Ixzi,Izyi:=Izyi]
```

$$Ii\_\_:=\begin{bmatrix} Ixxi & Ixyi & Ixzi \\ Iyxi & Iyyi & Iyzi \\ Izxi & Izyi & Izzi \end{bmatrix}$$

We enter also position vector $\vec{r}_i$, angular velocity $\vec{\omega}$, unit vector $\vec{n}$ (oriented as $\vec{\omega}$) and the distance R_i :

```
#5:   ri_ := [xi, yi, zi]
#6:   w_ := [wx, wy, wz]
#7:   n_ := w_/|w_|
#8:   Ri:=|CROSS(ri_,n_)|
```

Having done these preliminary steps we begin to prove the above relations.

a) We enter:

```
#9:  Ii:=n_`·Ii__·n_
#10: Ii-mi·Ri^2
```

and then simplify the expression #10

```
#11: 0
```

The result #11 ends the proof.

In a similar way we check: the relation for $\vec{L}_i$:

```
#12: Li_:=Ii__·w_
#13: vi_:=CROSS(w_,ri_)
#14: Li_-mi·CROSS(ri_,vi_)
#15: [0,0,0]
```

and the kinetic energy:

```
#16:  Eki := 1/2 · w_` · Ii__ · w_

#17:  Eki - 1/2 · mi · vi_ · vi_

#18:  0
```

The results #15 and #18 being the simplification of #14 and #17 respectively, confirm the validity of the examined formulae for any mass point m_i. A simple generalization proves their validity for the whole rigid body.

Comment: We can apply an alternative verification method of an equation. Instead of the expression of the form #10 we can enter the equation:

```
Ii = mi·Ri^2
```

Its simplification returns

```
true
```

A similar procedure can be applied to the kinetic energy

```
1/2 · mi · vi_^2 = Eki

true
```

Unfortunately this method is useless when comparing the vectors

```
Li_=mi·CROSS(ri_,vi_)
```

DERIVE does not display 'true' when the above equation is evaluated. In this case we evaluate the difference of the above vectors

```
Li_ - mi·CROSS(ri_, vi_)

                                  [0, 0, 0]
```

Perhaps the authors of DERIVE could remove this small inconvenience.

Principal axes

The principal axis is an axis of rotation with respect to which the angular momentum $\vec{L}$ has the same direction as the angular velocity $\vec{\omega}$. Mathematically it means that:

$$\vec{L} = \alpha\vec{\omega} \,, \qquad \alpha \text{ - scalar coefficient.}$$

On the other hand we have proved that for any rotation axis $\vec{L} = \hat{I}\vec{\omega}$. The above two relations lead to the eigenvalue problem:

$$\hat{I}\vec{\omega} = \alpha\vec{\omega} \quad \alpha \text{ - eigenvalue, } \vec{\omega} \text{ - eigenvector.}$$

After simple rearrangements one gets:

$\hat{I}\vec{\omega} = \alpha\vec{\omega} \;\; \Rightarrow \;\; \hat{I}\vec{\omega} = \alpha I\vec{\omega} \;\; \Rightarrow (\hat{I} - \alpha I)\vec{\omega} = \vec{0}$, where I - identity matrix.

The substitution of the form $\vec{\omega} = |\vec{\omega}|\vec{X}$) and elimination of $|\vec{\omega}|$ lead to the following factorized form:

$$(\hat{I} - \alpha I)\vec{X} = \vec{0} \;,$$

where $\vec{X}$ is a unit vector of the principal axis.

First we evaluate eigenvalues of the inertia tensor. These values can be obtained from the solution of the well known scalar equation

$$\left|\hat{I} - \alpha I\right| = 0$$

(the necessary condition for the existence of nontrivial solutions for eigenvectors)

Alternatively, by the use of the DERIVE function EIGENVALUES.

Example 1: We calculate orientations of the principal axes of the system of point masses m_1 , m_2 are located in the corners of the square of the side $2a$ (Fig. 2).

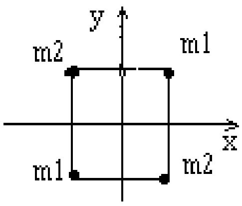

Fig.2. The configuration of the point masses of the system

Let us apply the computer algebra.

We enter:

a) the data for the problem, in the convenient vector notation (the way of entering the data is not unique):

m: = [m2,m1,m2,m1]

[x: = [a,a, – a, – a],y: = [a, – a, – a,a],z: = [0,0,0,0]]

n: = DIMENSiON(m)

b) the components of inertia tensor: (following page)

$$\left[Ixx:=\sum_{i=1}^{n} m_i\cdot\left(y_i^2+z_i^2\right), Iyy:=\sum_{i=1}^{n} m_i\cdot\left(x_i^2+z_i^2\right), Izz:=\sum_{i=1}^{n} m_i\cdot\left(x_i^2+y_i^2\right)\right]$$

$$Ixy:=\left(Iyx:=-\sum_{i=1}^{n} m_i\cdot x_i\cdot y_i\right)$$

$$Ixz:=\left(Izx:=-\sum_{i=1}^{n} m_i\cdot x_i\cdot z_i\right)$$

$$Iyz:=\left(Izy:=-\sum_{i=1}^{n} m_i\cdot y_i\cdot z_i\right)$$

c) the matrix form of the inertia tensor

$$I:=\begin{bmatrix} Ixx & Ixy & Ixz \\ Iyx & Iyy & Iyz \\ Izx & Izy & Izz \end{bmatrix}$$

We can now enter equation #14 and solve it:

#14: $DET(I - \alpha\cdot IDENTITY\_MATRIX(3)) = 0$

#15: $\alpha = 4\cdot a^2\cdot m2$

#16: $\alpha = 4\cdot a^2\cdot m1$

#17: $\alpha=4\cdot a^2\cdot(m1+m2)$

Inserting the eigenvalues that were found, one at a time, into the equation for the eigenvalue problem, we find eigenvectors (directions of the principal axis).

We assign names $\alpha 1$, $\alpha 2$ and $\alpha 3$ to the solutions:

#18: $\left[\alpha 1:=4\cdot a^2\cdot m2, \alpha 2:=4\cdot a^2\cdot m1, \alpha 3:=4\cdot a^2\cdot(m1+m2)\right]$

and evaluate the corresponding eigenvectors.

For $\alpha 1$ we have the equation #19 and its solution #20:

```
#19: (I - α1·IDENTITY_MATRIX(3)) · [X; Y; Z] = [0; 0; 0]

#20: [X=@1,Y=-@1,Z=0]
```

This result, #20, requires some comments. From the introduction to DERIVE we know that @ denotes any number. In our problem however *X, Y, Z* are the components of the unit vector (directional cosines) the squares of which should sum to one. The first solution is then:

$$X=\frac{\sqrt{2}}{2},\quad Y=-\frac{\sqrt{2}}{2},\quad Z=0.$$

In the similar way we evaluate the directions of the other principal axis:

```
#21: (I - α2·IDENTITY_MATRIX(3)) · [X; Y; Z] = [0; 0; 0]

#22: [X = @2, Y = @2, Z = 0]

#23: (I - α3·IDENTITY_MATRIX(3)) · [X; Y; Z] = [0; 0; 0]

#24: [X=0,Y=0,Z=@3]
```

The directions of all principal axes are then:

$$\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0\right),\quad \left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\quad (0,0,1)$$

As it was mentioned eigenvalues can be also obtained by using one of two DERIVE functions: EIGENVALUES or CHARPOLY. In particular if we enter #25 and simplify it we get #26

```
#25: EIGENVALUES(I,α)

#26: [α = 4·a^2·m2, α = 4·a^2·m1, α = 4·a^2·(m1 + m2)]
```

or if we enter the equation #28 and solve it then DERIVE returns

```
#28: CHARPOLY(I,α)=0

#29: α = 4·a^2·m2

#30: α = 4·a^2·m1

#31: α = 4·a^2·(m1 + m2)
```

Fourth International Derive TI-89/92 Conference

Not only eigenvalues, but also eigenvectors, can be evaluated by the use one of the DERIVE functions. We have at our disposal the function EXACT_EIGENVECTOR from the utility file VECTOR.MTH. Having loaded the file VECTOR.MTH we enter, one by one, the expressions:

EXACT_EIGENVECTOR(I, α1) =

EXACT_EIGENVECTOR(I, α2) =

EXACT_EIGENVECTOR(I, α3) =

and get

```
#32: EXACT_EIGENVECTOR(I, α1) = [ x1 = @5  x2 = -@5  x3 = 0 ]
#33: EXACT_EIGENVECTOR(I, α2) = [ x1 = @6  x2 = @6  x3 = 0 ]
#34: EXACT_EIGENVECTOR(I, α3) = [ x1 = 0  x2 = 0  x3 = @7 ]
```

The results #32, #33 and #34 are, as should be expected, the same as the former ones (#20, #22 and #24).

Example 2: In this example we demonstrate how to perform the calculations for a body with continuous mass distribution. As an example we consider a thin uniform square plate placed in the *xy* plane. The side of the plate is *a* and its mass *m*.

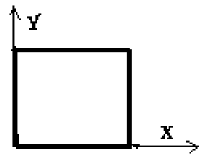

Fig.3 The uniform square plate with side *a* and mass *m*

We evaluate the following for the plate:

a) inertia tensor,

b) its eigenvalues,

c) principal axes,

d) the inertia moment with respect to the axis passing through the reference system center, whose inclination angles to the *x*, *y*, *z* axes are:

$\alpha = \pi/4,\ \beta = \pi/4,\ \gamma = \pi/2.$,

e) the angular momentum (components and value) with respect to the axis defined in (d), for a given angular velocity ω,

f) the rotational kinetic energy for the angular velocity $\vec{\omega} = [\omega_x, 0, 0]$.

To present variety of different approaches we will use another notation. If we introduce new variables $x = x_1,\ y = x_2,\ z = x_3$ then the components of inertia tensor are:

$$J_{\mu\nu} = \sum_{\alpha} \int_V \rho(\chi_\alpha \chi_\alpha \delta_{\mu\nu} - \chi_\mu \chi_\nu) dV = \int_V \rho(\chi_\alpha \chi_\alpha \delta_{\mu\nu} - \chi_\mu \chi_\nu) dV$$

where $\delta_{\mu\nu}$ denotes Koronecker's delta function. The notation assumes summation over indices that occur twice (standard notation).

Fourth International Derive TI-89/92 Conference

We enter the data of the problem

$$\#1: \quad \left[r := [x, y, z],\ \sigma := \frac{m}{a^2},\ z := 0 \right]$$

and the inertia tensor

$$I(k, l) := \int_0^a \int_0^a \sigma \cdot \left(\left(\sum_{i=1}^{3} r_i^{\ 2} \right) \cdot \mathrm{KRONECKER}(k, l) - r_k \cdot r_l \right) dx\ dy$$

```
#5:  I__:=VECTOR(VECTOR(I(i,j),j,1,3),i,1,3)
```

Note:

- The inertia tensor can be also evaluated with the help of the VOLUME_INERTIA function from the utility file INT_APPS.MTH.

a) Simplification of #5 returns the inertia tensor:

$$\#6: \quad I\_\_ = \begin{bmatrix} \frac{a^2 \cdot m}{3} & -\frac{a^2 \cdot m}{4} & 0 \\ -\frac{a^2 \cdot m}{4} & \frac{a^2 \cdot m}{3} & 0 \\ 0 & 0 & \frac{2 \cdot a^2 \cdot m}{3} \end{bmatrix}$$

b) Various methods of eigenvalue evaluation have already been presented. We will use one of them. Entering and simplifying the function:

EIGENVALUES(I__ , λ)

returns eigenvalues:

```
#7:  EIGENVALUES(I__,λ)
```

$$\#8: \quad \left[\lambda = \frac{2 \cdot a^2 \cdot m}{3},\ \lambda = \frac{a^2 \cdot m}{12},\ \lambda = \frac{7 \cdot a^2 \cdot m}{12} \right]$$

c) We assign the names λ1, λ2 and λ3 to the evaluated eigenvalues, load the utility file VECTOR.MTH and evaluate, one by one, the corresponding eigenvectors:

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```
#9:  [λ1:=2·a^2·m/3, λ2:=a^2·m/12, λ3:=7·a^2·m/12]

#10: EXACT_EIGENVECTOR(I__,λ1)

#11: [ x1=0   x2=0   x3=@1 ]

#12: EXACT_EIGENVECTOR(I__,λ2)

#13: [ x1=@2   x2=@2   x3=0 ]

#14: EXACT_EIGENVECTOR(I__,λ3)

#15: [ x1=@3   x2=-@3   x3=0 ]
```

Remembering that the symbol denotes 'any number' and that the sum of squares of directional cosines is equal to one, we can conclude from #11, #13 and #15 that:

-one of the principal axes is perpendicular to the plate i.e. $\gamma = 0$

-the other ones are identical with diagonals of the square plate i.e. $\alpha = \dfrac{\pi}{4}, \beta = \dfrac{\pi}{4}, \gamma = 0$, and $\alpha = 3\pi/4, \beta = \pi/4, \gamma = \pi/2$.

d) It was shown in the introduction that inertial moment of a rigid body is $J_w = \vec{n}^T \hat{J} \vec{n}$, where: $\vec{n} = (\cos\alpha,\ \cos\beta,\ \cos\gamma)$ - the unit vector for an axis of rotation, $\hat{I}$ - inertia tensor.

We enter the above expressions and data:

```
#16: n_ := [COS(α); COS(β); COS(γ)]

#17: [α := π/4, β := π/4, γ := π/2]

#18: Iner = n_` · I__ · n_
```

after #18 is simplified one gets

```
#19: Iner=[a^2·m/12]
```

e) We have to enter: the angular velocity vector, the angular momentum vector (see problem

```
#20: w_ := [wx; 0; 0]

#21: L_ := I__ · w_
```

and evaluate it:

$$L\_=\left[\begin{array}{c} a^2\cdot m\cdot wx \\ \dfrac{a^2\cdot m\cdot wx}{4} \\ 0 \end{array}\right]$$

By comparing #20 with the obtained result we observe that the directions of $\vec{\omega}$ and $\vec{L}$ vectors differ.

f) The kinetic energy of rotational motion is defined by #24 which, when simplified returns the result #25:

$$\#24:\ Ek = \frac{1}{2}\cdot w\_` \cdot I\_\_ \cdot w\_$$

$$\#25:\ Ek=\left[\frac{a^2\cdot |m\cdot wx|}{6}\right]$$

Remarks: The function EXACT_EIGENVECTOR appears to be better than the SOLVE function. To show this let us calculate eigenvectors of the following matrix

$$\begin{bmatrix} 2\cdot A & 0 & 0 \\ 0 & A & -B \\ 0 & -B & A \end{bmatrix}$$

One of its eigenvalues is 2A (see below) . We get the corresponding eigenvector with the help of the function EXACT_EIGENVECTOR

$$\text{EIGENVALUES}\left(\begin{bmatrix} 2\cdot A & 0 & 0 \\ 0 & A & -B \\ 0 & -B & A \end{bmatrix}, \lambda\right) = [\lambda=A+B, \lambda=A-B, \lambda=2\cdot A]$$

$$\text{EXACT\_EIGENVECTOR}\left(\begin{bmatrix} 2\cdot A & 0 & 0 \\ 0 & A & -B \\ 0 & -B & A \end{bmatrix}, 2\cdot A\right)$$

$$[\ x1=@1 \quad x2=0 \quad x3=0\ ]$$

However if we use SOLVE function we get no solution

$$\text{SOLVE}\left(\begin{bmatrix} 2\cdot A & 0 & 0 \\ 0 & A & -B \\ 0 & -B & A \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\cdot A\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

$$[]$$

To overcome this shortcoming we simplify the system of equations

$$[\text{true}, [A\cdot Y-B\cdot Z=2\cdot A\cdot Y], [A\cdot Z-B\cdot Y=2\cdot A\cdot Z]]$$

It can be easily seen that first the equation of the above system is just the identity (‘true’). In such situation we solve the system consisting of the remaining two equations and we get.

$$\text{SOLVE}\left(\begin{bmatrix} A\cdot Y-B\cdot Z=2\cdot A\cdot Y \\ A\cdot Z-B\cdot Y=2\cdot A\cdot Z \end{bmatrix}, \begin{bmatrix} Y \\ Z \end{bmatrix}\right)$$

$$[\ Y=0 \quad Z=0\ ]$$

Conclusion

DERIVE is a satisfactory and useful tool for teaching rigid body physics.