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Solving Max-Min and Related Rate Problems Empirically using Geometry on a TI-92 and then Analytically using Scripts

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I started developing these geometry applications because I don't feel students really understand what it means to find the rectangle of a constant perimeter that has maximum area. When I saw what the geometry application could do, I realized I could animate the above problem. Then I began developing the applied max-min problems I will show today

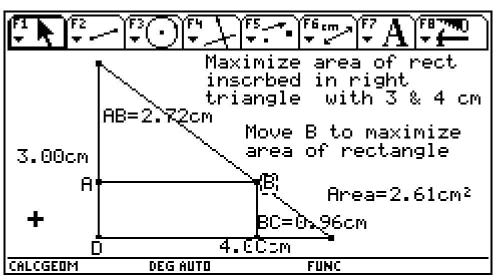
maximizing the area of a rectangle inscribed in a right triangle,
forming a square and a circle from a piece of wire,
minimizing the surface area **and** cost of an open box,

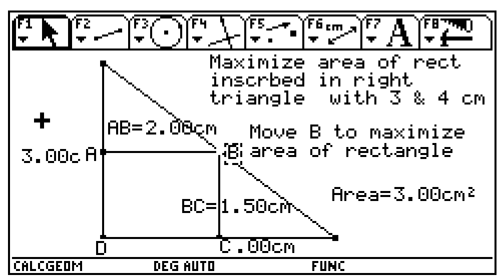
and the related rate problems

growth of a ripple, and
observing a ladder sliding down a wall.

Maximizing the area of a rectangle inscribed in a right triangle (APPS 8: recttri, 9: rectrisc)

A right triangle has legs 3 cm and 4 cm. Find the rectangle with maximum area that is inscribed in the right triangle with one vertex on its hypotenuse and two sides along its legs.





To open this application (if it is on your calculator), press the blue **APPS** button**,**
8:Geometry **2:Open** **Folder: calcgeom** down arrow to **Variable:**
right and down arrow to **recttri** **ENTER** **ENTER**

When you have a screen similar to the ones above, press **ESCAPE**, move the cursor to point **B**, when the screen says **THIS POINT** press the hand on the blue key in the upper left of the TI-92. When you see the hand grab the point, left or right arrow to move **B** along the hypotenuse of the triangle. Watch how the area changes. This clearly shows that there are infinitely many rectangles inscribed in this right triangle and that they have different areas Thus it makes sense to try to find the rectangle with maximum area. Move **B** to maximize this area as in the right screen above. We have found the solution empirically. It is interesting to note that the ratio of the sides of this rectangle of maximum area (1.5:2) equals the ratio of the legs of the right triangle (3:4). Would this be true for the rectangle of maximum

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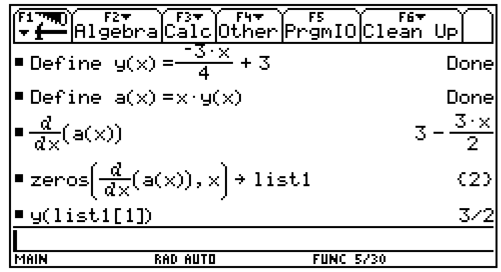
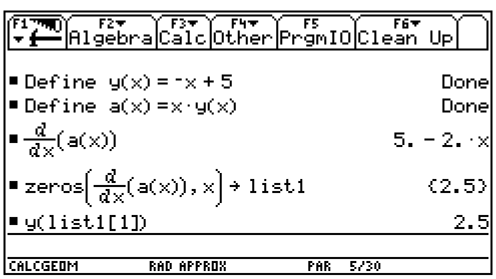
area inscribed in any right triangle? To answer this question more generally and to see the analytical solution, we can use a script and CAS. To open the script (if it is on your calculator),

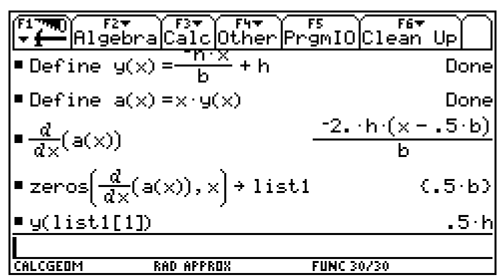
press the blue **APPS** button**, 9:Text Editor 2:Open Folder: calcgeom** down arrow to **Variable:** right and down arrow to **rectrisc ENTER ENTER**

:Maximize the area of a rectangle inscribed in a right triangle with legs 3 and 4 cm with one vertex on the hypotenuse and with two sides on the legs of the triangle.
:The point B with coordinates (x,y) lies on the line through (4,0) and (0,3). Thus its slope is ª3/4 and its equation is y = ª3x/4 +3.
:We need to write the area of the rectangle as a function of x and find where its derivative is 0.
C:define y(x) = ª3x/4 + 3
C:define a(x)=x*y(x)
C:¶(a(x),x)
C:zeros(¶(a(x),x),x)»list1
C:y(list1[1])
:Thus the rectangle of maximum area has dimensions 2 cm x 1.5 cm.



This script describes the problem, contains the command lines (with C) to solve this problem analytically, and summarizes the result. The screen at right shows the CAS solution. To solve the problem for different right triangles, simply change the equation of the hypotenuse and execute it again. For a right triangle with legs 5 cm and 5 cm, the line is y = ˉx + 5.





Again, we see the ratio of the sides of the rectangle of maximum area equals the ratio of the legs of the right triangle (5:5 = 2.5:2.5). The power of CAS is that we can also check this for a right triangle with base b and height h; the line is y = -h * x / b + h and we see that our conjecture is true in general: b:h = .5b:.5h.

To construct this application, press the blue **APPS** button**, 8:Geometry 3:New**
Folder: calcgeom name the **Variable: recttri2 ENTER ENTER**

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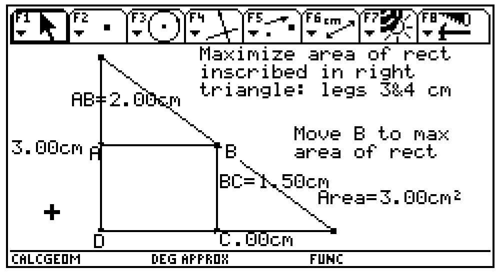
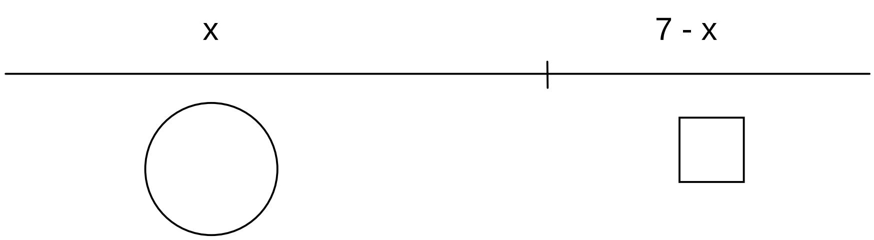
Create a vertical segment [F2, 5] starting near the screen's upper left, print its length [F6, 1], adjust the segment to be exactly 3.00 cm [using ESC and hand], and label its lower endpoint **D** [F7, 4] . Then construct a line perpendicular to it through **D** [F4, 1], create a segment on this line [F2, 5] and adjust it to be exactly 4.00 cm [using hand]. Hide the line [F7, 1]. Create the segment that is the hypotenuse [F2, 5]. Place a point **B** on the hypotenuse [F2, 2]. Draw lines



through **B** perpendicular to each leg [F4, 1], find the intersections of these perpendiculars with the legs [F2, 3], label those points **A** and **C**, and hide the perpendiculars [F7, 1]. Create the polygon **ABCD** [F3, 4]. Print the area of this rectangle [F6, 2] and move it to the lower right part of the screen [ESC hand]. Create the segments **AB** and **BC** [F2, 5] and print their lengths [F6, 1]. Then enter the instruction comments if you wish [F7, 5].

Forming a square and a circle from a piece of wire (APPS 8: wire, 9: wirescr)

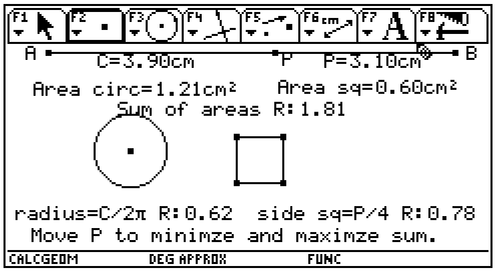
One of my favorite problems conceptually is to cut a wire of 7 cm into two pieces, fold one piece into a circle and the other into a square, and ask how to cut the wire to minimize the sum of the areas; then to maximize the sum. The solution of the problem analytically is difficult for most Calculus I students. The geometry application allows us to see the solution empirically; CAS lets us find the analytical solutions easily and concentrate on writing the sum function.

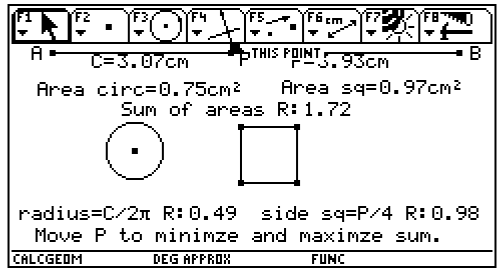


$C = 2\,\pi\, r = x$ $\qquad$ $P = 4\,s = 7 - x$

Thus $\quad A_{circle} = \pi r^2 = \pi\left(\dfrac{x}{2\pi}\right)^2 = \dfrac{x^2}{4\pi}$ and $A_{square} = s^2 = \left(\dfrac{(7-x)}{4}\right)^2 = \dfrac{(7-x)^2}{16}$.

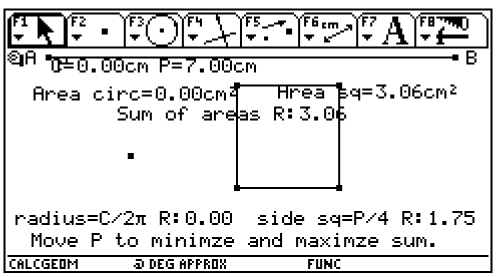
We wish to minimize and maximize $A_{total} = \frac{x^2}{4\pi} + \frac{(7-x)^2}{16}$ on the interval $0 \le x \le 7$.

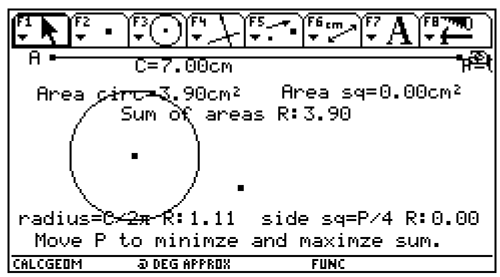




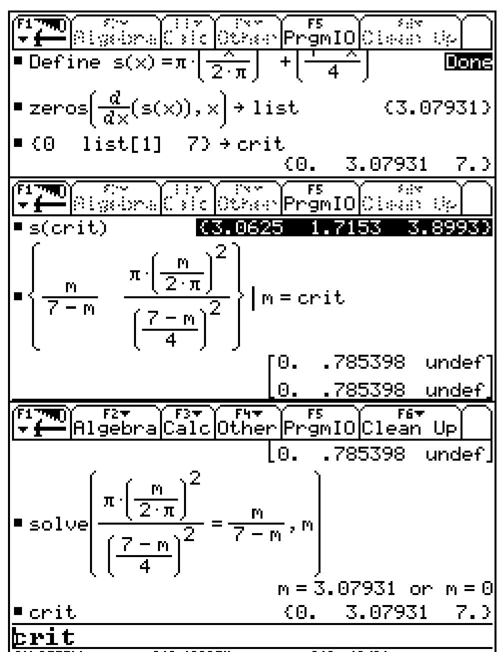
Open the geometry application **wire** (if it is on your calculator). Move P along the wire at the top of the screen and observe the sum of the areas of the circle and square. The minimum sum is 1.72cm^2 with the wire cut almost in half. Is their any way to understand why this particular subdivision of the wire minimizes the sum of the areas? Let's examine the ratio of pieces of the wire and the ratio of the areas of the figures. 3.07/3.93 ≈ 0.75/0.97.

The sum of the areas increases as we move toward either endpoint. Thus the maximum area, 3.90 cm^2, is an absolute maximum and occurs if the whole wire forms a circle (x = 7) which the geometry application shows very clearly. (Be careful not to release the hand at an endpoint.)





Open the script **wirescr** (if it is on your calculator).



:A wire 7 cm long is divided into two pieces. One piece
is bent to form a circle; the other to form a square.
How should the wire be split to
a) minimize the sum of the areas of the circle and the
square
b) maximize the sum of their areas?
C:Define S(x) = Œ*(x/(2Œ))^2+((7-x)/4)^2
C:zeros(¶(S(x),x),x)»list
C:{0,list[1],7}»crit
C:S(crit)

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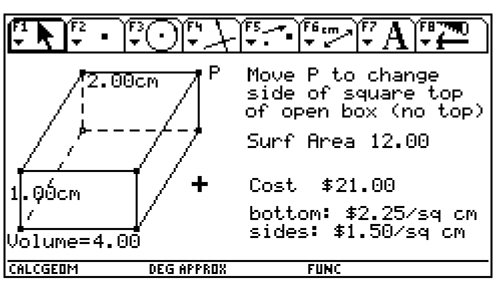
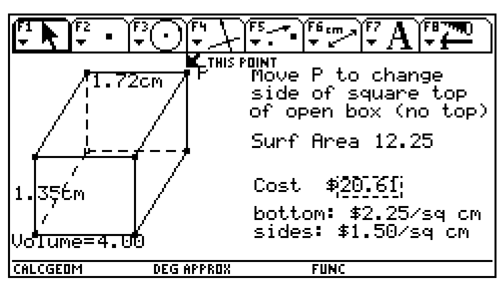
:Thus the maximum total area occurs   when all the
 wire is formed into a   ______?
:To try to make sense of the answer for the minimum sum
of the areas, Calculate the ratio of the lengths
of the sides and the ratio of their areas at this
minimum value.
C:{m/(7-m),Œ(m/(2Œ))^2/((7-m)/4)^2}|m=crit
:Wow, they are equal! Are these ratios always equal or
only at this point?
C:solve(Œ*(m/(2*Œ))^2/((7-m)/4)^2=m/(7-m),m)
C:crit
:The ratios of the areas and the sides are equal or
undefined (m=7) at the critical points for the sum
of the areas!   WHY?

Although finding the minimum and maximum values of the sum of the areas analytically is complicated, we can find them easily by using CAS. The zero of the derivative is 3.079. Calculating the sum of the areas for this value and at the endpoints (x = 0, 3.079, 7), we see in the upper right screen that the minimum value occurs at the interior point and the maximum value at the right endpoint. When x = 7, the whole wire is formed into a circle. Calculating the values at which the ratio of the areas of the figures equals the ratio of the subsegments of the wire, we find in the lower right screen that these values are critical values of the sum of the areas. We can show that this result generalizes by replacing the constant 7 by h in S(x) and the solve command near the end of the script above.

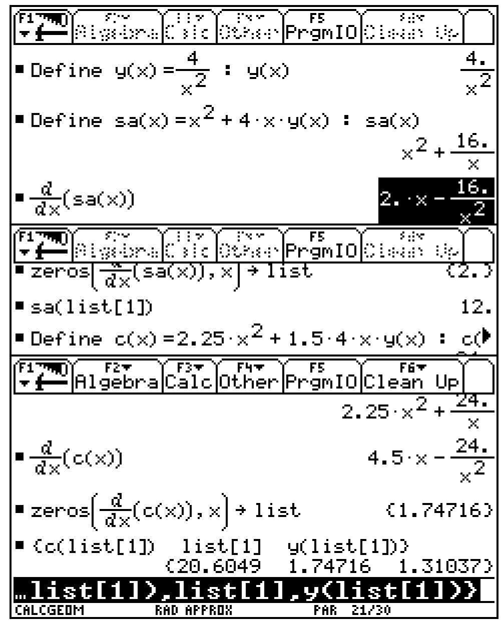
Minimizing surface area and cost of an open box (APPS 8: openbox, 9: mincost)

Find the dimensions of an open box (no top) with a square base and a volume of 4 cm^3 with minimum surface area. Open the geometry application **openbox**. Move P to minimize the surface area (12.00 cm^3). If the material used to make the base costs \$2.25/cm^2 and the material for the sides \$1.50/cm^2, this box with minimum surface area costs \$21.00. Moving P further we see that the minimum cost for this box is slightly less, \$20.61, and see approximate dimensions of the box. Note how this one application shows that the box with minimum surface area and the box with minimum cost have different dimensions.





To solve this problem analytically, $x^2 * y = 4$ or $y = 4/x^2$.
Thus the surface area is $x^2 + 4x*y = x^2 + 16/x$.
Its cost is $2.25 x^2 + 1.5*4x*y = 2.25 x^2 + 24/x$.
To minimize the surface area and cost using CAS, open the script **mincost**.

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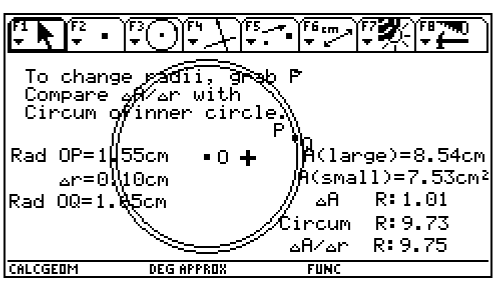
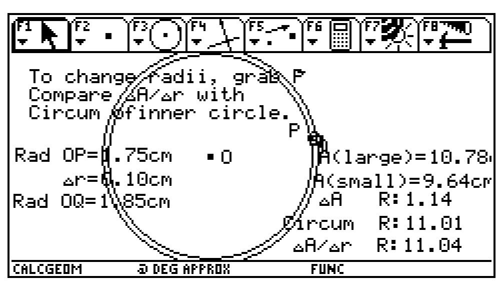
:Minimize the surface area of an open box (no top)
  with constant volume 4 cu cm.
  Vertical side y=4/x^2
C:define y(x) = 4/x^2:y(x)
C:define sa(x)=x^2+4x*y(x):sa(x)
C:¶(sa(x),x)
C:zeros(¶(sa(x),x),x)»list
C:sa(list[1])
 :
 :Then minimize its cost if the base costs $2.25/sq
    cm and the sides cost  $1.50/sq cm .
C:define C(x)=2.25*x^2+1.50*4*x*y(x):C(x)
C:¶(C(x),x)
C:zeros(¶(C(x),x),x)»list
C:{c(list[1]),list[1],y(list[1])}
 :The open box with V=4 has min cost of $20.61 if
    its dimensions are   1.75 cm x 1.75 cm x 1.31
    cm.
```

Related rates: growth of a ripple (APPS 8: ripple)

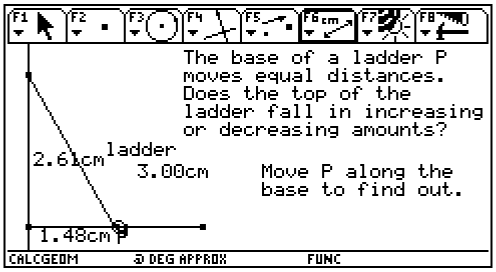
A stone is thrown into a still pond. Find the rate at which the ripple is changing when the radius is 1.55 cm or 1.75 cm. Open the geometry application **ripple**. Notice that the change in area with respect to the original radius is very close to the circumference of the circle.

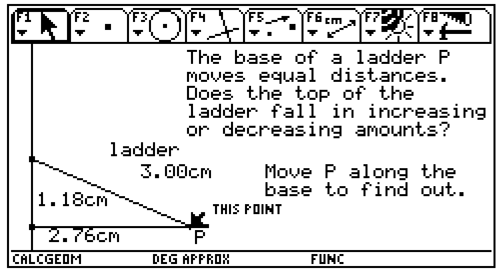




Observing a ladder sliding down a wall (APPS 8: laddergm, 9: ladder, 9: laddcalc)

The base of a ladder of length 3 meters is pulled away from a wall in equal increments.

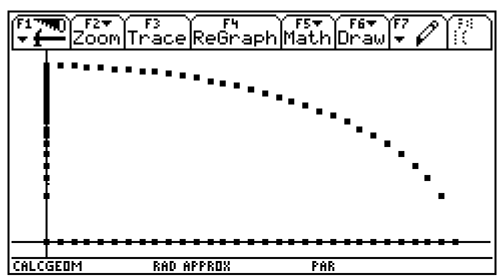




Open the geometry application **laddergm**. Move P to observe the rate at which the ladder slides down the wall; is it sliding at a constant rate or does it appear to be sliding faster near the bottom of the wall? Open the script **ladder** to see a different representation of this motion (based on a script written by Dr. Paul Beem, Indiana University at South Bend). Let **t** represent the distance of the base of the ladder from the wall and **s(t)** = $\sqrt{(3^2 - t^2)}$ the height of the ladder on the wall. Graph the functions xt1(t) =t, y1(t) = 0, x2(t) = 0, y2(t) = s(t), x3(t) = t, y3(t) = s(t) in the parametric window [0,3] x [-1, 4] x [-1, 4] with t-step = 0.2. The first (horizontal) function gives the positions of the base of the ladder along the ground (equal increments), the second (vertical) function the positions of the top of the ladder sliding along the wall, the third (2-d) function the height on the wall versus the distance along the ground. The second and third functions show that the changes in height are greater as the ladder moves away from the wall.

:A ladder is pulled away from a vertical wall at the rate of .1 ft/second. Does the rate at which it falls increase or decrease as the ladder is pulled farther from the wall?
C:Define s(t)=§(3^2-t^2)
C:setMode("Graph ","Parametric"):setGraph("Graph

Order","Simul"):animate():setpwind(0,3,ª.25,3 .25,ª.25,3.25)
C:Define xt1(t)=t:Define yt1(t)=0:Style 1,"square"
C:Define xt2(t)=0:Define yt2(t)=s(t):Style 2,"square"



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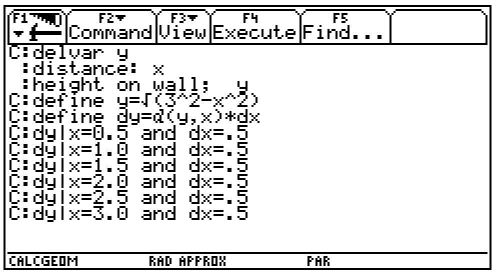
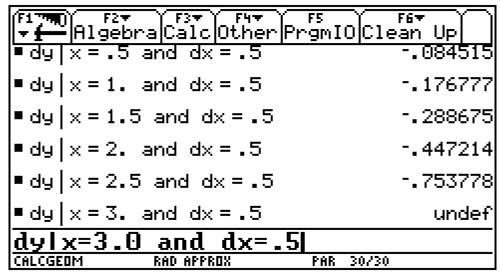
C:Define xt3(t)=t:Define yt3(t)=s(t):Style 3,"square":.1»tstep **(cont..)**
C:DispGr
:How fast is the ladder sliding down when it is 2 ft from the wall?
C:Define y(x)=§(9-x^2):y(x)
C:Define dy = ¶(y(t),t)*dt:dy
:

To see the changes in height, use the script below, **laddcalc**, to see the differences in the heights of the ladder as it is pulled away from the wall. The negative signs show that the ladder is falling; the increasing absolute values of dy show that it is falling faster as the ladder is pulled away from the wall.





The more I use scripts the more I realize their power. Why use scripts instead of writing a program? First, scripts use regular TI-92 commands, not special programming commands and logic. Secondly, scripts let us see the result of each command, not just a final result emerging from a "black box." Thus, they are much more accessible for students to use without needing to learn a new language. Thirdly, a script can be changed easily to solve related problems and to generalize results. Instructors can write scripts and link them to students to use as worksheets. Students can save their work as lab reports and print them out to turn in by using TI's GraphLink software.