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The Logarithmic Function

Jasna Kos
Gimnazija Belligrad
Slovenia
Email: jasna@s-gimb.lj.edus.si

Matija Lokar
SloveniaUniversity of Ljubljana
Slovenia
Email: Matija.Lokar@fmf.uni-lj.si

Introduction

The ways in which the logarithmic function is taught differ. According to the curriculum in Slovenia, logarithmic function is introduced as the inverse function of exponential function in the second year of high school (10^{th} year of schooling) to 16 year old students.

The whole aspect of the subject is presented - from the introduction to the usage of logarithmic function in this workshop. We would like to show different approaches to teaching the topic. So the classical way of teaching where teacher teaches and the mathematics grows in front of students' eyes is combined with new possibilities the technology offers. Therefore we use videos, computer prepared lectures and Internet as well as classical pictures and perhaps even fresh vegetables like cauliflower. We would like to emphasise the importance of the self-discovering of a new topic, where the student, with the aid of computer, learns a new mathematical fact by himself. Of course this should be supported with carefully and systematically prepared lessons by the teacher. As has been shown in several studies students usually remember what they have learned by this method very well. Active participation in discovering mathematics makes the students feel successful. Consequently, they develop a much better attitude towards mathematics. We have found out that students, who are not forced to study certain topics in mathematics in the classical way of learning, but have discovered the subject with the aid of a computer, want to get more details about that topic. They follow the teacher's explanations, derivations and proofs much more easily. The best lessons are the ones where our students and we do not think about the technology being used any more, so teaching and learning with technology becomes just teaching and learning.

In this workshop, we will present several examples of teaching materials we have developed. Our intention is that each participant becomes a student again. After a brief introduction of the tasks, the main part of the workshop will be devoted to solving the tasks in the way students solve them. In the last part of our workshop we will try to encourage a discussion about the materials presented and the experience participants have with teaching logarithmic function.
As time devoted to various aspects of logarithmic function as well as teaching styles differ, all presented worksheets may be used independently. They are also meant merely as a basis for the teachers' own work, so it is very beneficial if they are available in electronic form. All worksheets can be found on http://rc.fmf.uni-lj.si/matija/logarithm/logfun.htm. Throughout the paper we mention a lot of WWW pages connected tightly or just briefly with the subject being discussed.

Links are used for two reasons: as starting points for students who would like to get deeper into the subject and also as a source of ideas for teachers to develop suitable worksheets.

Discovering a new mathematical subject described above takes more time than classical lessons. The advantage is that students with their intuition, prepared worksheets that guide them through the subject, and data obtained from the Internet learn new things. The knowledge gained on the basis of self discovery and experience is usually deeper and more lasting. The explained method of teaching encourages communication between students and the teacher as well as between the students themselves. The classical way of teaching often lacks this type communication. Students are forced to use mathematical terminology, they have to explain their ideas more precisely. We have also observed a substantial increase in their ability to express themselves mathematically. We must not neglect the classical way of teaching. There is always a group of students in the classroom which needs additional explanations, guidance, … The teacher's role is especially important at the end when he has to systematically summarise the lessons and review the subject with the carefully prepared questions.

The Rules of Logarithmic Computation

In the first worksheet the rules of logarithmic computation are introduced. With the aid of a computer algebra system like *DERIVE*, students try to discover the three main rules of logarithmic computation, namely addition and subtraction of logarithms and calculating logarithms of the power function.

$$\ln x + \ln y = \ln(xy)$$

$$\ln x - \ln y = \ln \frac{x}{y}$$

$$\ln x^n = n \cdot \ln x$$

We should also warn the students that in *DERIVE* log (x) does not denote a common logarithm or a Briggsian one as it is usual. Logarithmic function with the basis a is written as *log (x, a)* and logarithmic function with the basis e with log x or ln x. As *DERIVE* transforms logarithms with other bases to natural logarithms and thus prevents us from building up too many rules, natural logarithms are used in the worksheet.

We do not expect students to have any difficulties with the mathematical notation of rules, but more problems will probably occur when the rules have to be expressed with words and suitable sentences made. We think this part is the important one; so we should insist in solving this part too, regardless of the time spend on clarifying this question. The whole exercise is quite short, so a substantial amount of time can be devoted to discussion. We also direct students to check various WEB resources devoted to logarithmic function, such as:

On the history of the logarithmic function:

- `http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Napier.html`
- `http://www.sosmath.com/algebra/logs/log1/log1.html`
- `http://britannica.com/bcom/eb/article/8/0,5716,118178+3,00.html`
- `http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Briggs.html`

On Eulers' number e and the natural logarithm

- `http://mathforum.com/dr.math/faq/faq.e.html`
- `http://www.math.utoronto.ca/mathnet/answers/ereal.html`

- http://duke.usask.ca/~fowlerr/e.html

On rules for calculating logarithms

- http://www.sosmath.com/algebra/logs/log4/log41/log41.html
- http://www.shodor.org/UNChem/math/logs/index.html
- http://www.physics.uoguelph.ca/tutorials/LOG/index.html
- http://www.math.utah.edu/~alfeld/math/log.html
- http://www.cne.gmu.edu/modules/dau/algebra/exponents/lexercises_frm.html
- http://taipan.nmsu.edu/aght/soils/soil_physics/tutorials/log/log_home.html

and many more, especially on Ask Dr. Math:

http://forum.swarthmore.edu/dr.math/tocs/logarithm.high.html

In the next lesson the classical way of teaching is assumed to be used where the rules that the students had discovered before are also proven in a mathematically correct way.

For solving the exercise we assume no prior knowledge of the *DERIVE* program, so all necessary steps with entering the expressions and similar are described. Since all computations will be done with natural numbers only, *DERIVE* should be set appropriately, namely so that all variables are assumed to be natural numbers.

At the moment most of our schools have *DERIVE FOR WINDOWS VERSION 4*, but as the new version, Version 5, seems to be much more appropriate for school use, both versions are explained whenever necessary.
The electronic form of the students' worksheet can be found on

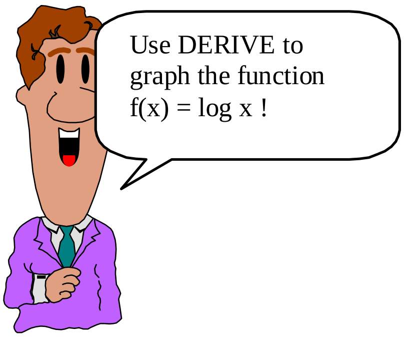
http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/rules.htm

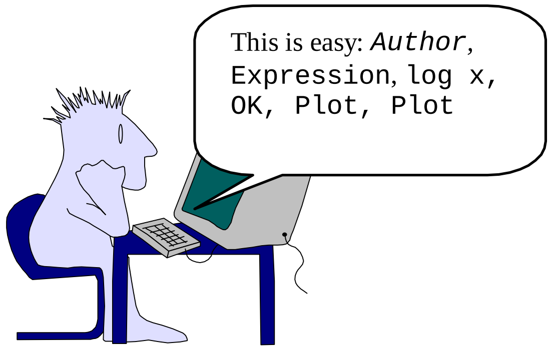
The Logarithmic Function – translation and scaling

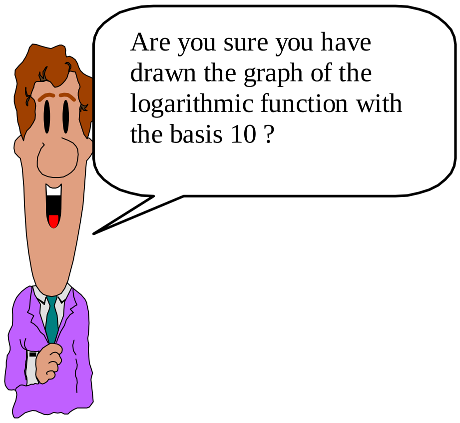
In the second worksheet the graph of the logarithmic function, is presented *DERIVE* will be used to find out how translation and scaling affect the graphs of logarithmic functions. This task can be used with students already familiar with the basic graphs of logarithmic function.
Again in the introduction we briefly describe the necessary commands required to accomplish the tasks. We assume students already have some knowledge about the program, so the explanations are quite brief, intended just for those not so proficient in *DERIVE*. The teacher's introduction is still necessary as the function log x in *DERIVE* is not the common logarithm. The experience one teacher had when he (actually she) asked the students to graph the function log(x) and had of course the common logarithm in mind is presented in the cartoon on the next page.

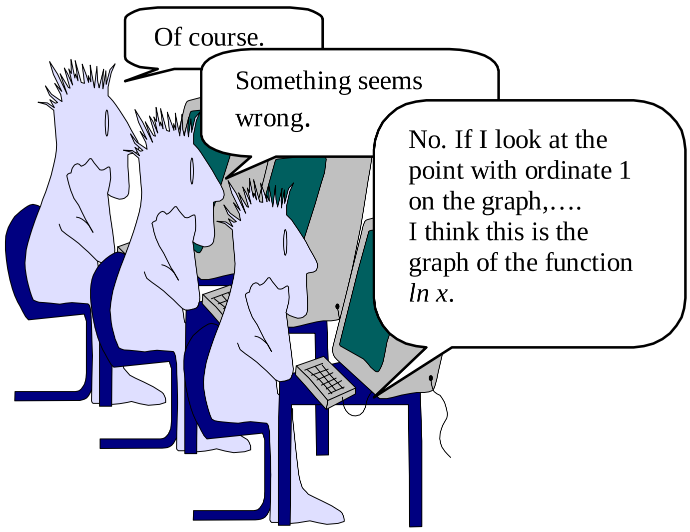
The exercise can be done in different settings regarding the time available and how familiar students are with *DERIVE*. Students can merely observe the teacher's presentation and contribute their suggestions. If they are already more familiar with the program, they can fill in the worksheets in the computer lab or as part of their home assignments. Of course in the classroom a detailed analysis of their work has to be done.

What's the exercise about? The students draw families of functions $f(x)=\ln(x-a)$, $f(x)=a\log_5(x)$ and $f(x)=\log_a(x)$. They try to describe the effect of parameter *a*. We expect them to get a feeling for what the graph of the function in the form $f(x)=a\log_c(x-b)$ looks like.









In the next lesson we will cover the subject and state the findings, clearly subsequent lessons are then carried on classically, as according to our curriculum students must be able to draw graphs of the functions without the computer or even graphical calculators. The computer is only used just to check the graphs.

The electronic form of the worksheet can be found on
http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/graph.htm

Solutions to Equations and Inequalities with the Logarithmic Function

Most exercises in Slovene textbooks only discuss equations and inequalities that can be solved elementarily. There are few examples (marked as harder, so usually avoided by students and teachers as well) where, for example, graphical methods have to be used to solve them. As computer algebra systems help us to use this different approach too, the main emphasis in this exercise is on solving the equations graphically. So in the worksheet a detailed approach to graphical solutions based on an example is presented.

The electronic form of the students' worksheet can be found on
http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/solve.htm

Using the Logarithmic Function

Introduction to this exercise again starts with a warning about the atypical syntax of log function in *DERIVE*. The exercise is presented to students who are already familiar with various aspects of logarithmic function. In this exercise we take a look at some "real life" examples where logarithmic function is used. They are from the fields not so often used in mathematics, namely, they come from music and psychology. We talk about memory curves and musical scales. One real life example comes, as usually from physics, namely the intensity of sound is discussed. In the last example we enter a field of more modern mathematics. Students often hear about fractals in art and design. So it is almost necessary to introduce the mathematical view of this topic.
Most of the students are capable to solve two or more exercises within one school lesson. However, to allow easier combining and perhaps also to prepare some more examples in the future, the introduction and each of the exercises are separated. The class can also be divided into groups. Each group can work on a different problem and then summarise its findings for the whole class.
Here WWW again moves to be a valuable tool. But we should be careful how much time is devoted to "surfing on the net" as is it to easy to slip to http://www.nba.com or similar, more interesting pages. Some examples of web pages besides those already mentioned each exercise:

- Rocket equations (http://www.execpc.com/~culp/rockets/rckt_eqn.html) with calculating how high the rocket model will go (http://www.execpc.com/~culp/rockets/qref.html)
- Exercises in Math Readiness on http://math.usask.ca/readin/menu.html with number of exercises where log function is needed http://math.usask.ca/readin/examples/expgrtheg.html
- A common example of exponential decay is *radioactive decay*. http://math.usask.ca/readin/examples/expdeceg.html
- On earthquakes and Richter Magnitude http://www.seismo.unr.edu/ftp/pub/louie/class/100/magnitude.html
- Many different problems: http://forum.swarthmore.edu/dr.math/tocs/logarithm.high.html

Lessons do not require any special knowledge of *DERIVE*, but it is assumed that students have already worked with the program. In the introduction the necessary commands are briefly mentioned. The following commands are used:

- To write expressions: `Author/Expression`;
- To open a new plot window: `Window/New 2D Plot`;
- To draw functions: switch to 2D Plot window. There the command `Plot` (*DERIVE FOR WINDOWS 4*) or `Insert/Plot` (*DERIVE FOR WINDOWS 5*) is used;
- In order to switch between a plot and an algebra window we have to choose the command Window and then choose one of the windows given below;
- To change the unit size on co-ordinate axes: `Set/Scale` in version 4, and `Set/Plot Region` in version 5.
- To approximate the value of expressions: `Simplify/Approximate`, and the `Approximate` button
- To solve equations: `Solve/Algebraically`, the `Simplify` button in version 4 and `Solve/Expression`, the `Simplify` button in version 5.

However, when we used the lesson Mathematics and Music in the first form in which the regression curve was explicitly given, we observed students asking themselves where the curve representing data comes from. Of course there is no time to discuss the least squares method in details. But to give the students some hands on experience, we decided to prepare suitable Utility function, so that the students produce the function by themselves. Therefore instructions for using utility functions were necessary. They are also needed when we are working with fractals.
The electronic form of the students' worksheet can be found on
http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/usage.htm

Using the Logarithmic Function in Psychology

The logarithmic function can be found in psychology as well. An example of logarithmic function is the curve of forgetting which is represented by the graph of the following function: $f(t) = A - B \log(t+1)$, with A and B being experimentally determined constants. The parameter t is time that has lapsed from the moment we have learned something, expressed in months, and f(t) is the result of a test that measures your knowledge, expressed in percentage terms.

In this lesson we use this simple mathematical model to answer various questions. We also show the weaknesses of such a simple model but on the other hand even with this model we can get some meaningful results. Usually quite a lively discussion occurs in the classroom and perhaps these students will never ask the question that is so often heard "Why do I have to learn about logarithmic function?" Namely, besides the usual questions about the knowledge of the students observed in the experiment we also ask when the gained knowledge of the observed group will drop to 0 or even below that. With a very simple example we show that we should always check the relevance of the answer as well as the relevance of the question itself. The worksheet can be found on

http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/psycho.htm

Mathematics and Music

In this exercise we discuss the connections between frequency, hearing, C-minor scale, octaves, piano keyboard and similar. We all know that the human ear can transform the undulation of air into sound in the range from 16 to 20000 Hz. The C-minor scale is part of the West-European tonal range, in which music has been written for more than three hundred years.

First we take a look at the table of approximate frequencies (f) for the tones in the first octave. X denotes the distance of the tone from the first C-tone. One octave is represented by tones that are 12 units apart. With the data for the first octave, we would like to make the function, representing the data and use it to calculate the width of a piano keyboard that could play all the tones in the human hearing range. We do not go into details, we just mention that in *DERIVE* a special method is implemented, called the least squares method. This is a mathematical procedure for finding the curve fitting best to a given set of points by minimising the sum of the squares of the offsets (``the residuals") of the points from the curve. The explanation, which can be found on various WWW sites, such as

- http://www.rci.rutgers.edu/~gea/notes/least.html,
- http://www.shodor.org/UNChem/math/lls/index.html,

can be valuable to more eager students. We can also use other tools like TI-89 which have the regression curve incorporated in the menu.

Due to interest in representing data with various curves, we plan to make a special worksheet devoted only to implementation of the Least squares method. Namely, to obtain various families of curves we have to change the data properly – again an application of the logarithmic function.

If we only use this exercise in a lecture we have plenty of time for discussion. So we can explore the connections between mathematics and music a little more. Some starting points:

- http://forum.swarthmore.edu/dr.math/problems/questkid9.27.98.html
- http://forum.swarthmore.edu/dr.math/problems/benson2.10.97.html
- http://www.math.niu.edu/~rusin/uses-math/music/
- http://www.sciencenews.org/sn_arc98/6_20_98/mathland.htm
- http://www.utm.edu/research/primes/programs/music/listen/
- http://www.utm.edu/~caldwell/midi/primes.cgi

The whole worksheet is on
http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/music.htm

Loudness of Sound

We use a logarithmic scale when there is a wide range of values, and when the change in a value depends not on the absolute size of the change but on the proportion to the value itself. The same is tone of the decibel scale. There are two reasons why a logarithmic scale is useful:

Quantities of interest exhibit such ranges of variation that a dB scale is more convenient than a linear scale. For example, sound pressure radiated by a submarine may vary by eight orders of magnitude depending on direction.

The human ear interprets changes in loudness within a logarithmic scale. The sensation of loudness of sound is not proportional to the energy intensity, but is rather a logarithmic function. *Loudness,* in Bels (after Alexander Bell), of a sound of intensity I is defined to be $L = \log \frac{I}{I_0}$, where I_0 is the minimum intensity detectable by the human ear (such as the tick of a watch at 6 m under quiet conditions), $I_0 = 10^{-12} W/m^2$. When a sound is 10 times as intense as another one, its loudness is 1 Bel greater. If a sound is 100 times as intense, it is louder by 2 Bells, and so on. A bell is a large unit, so a sub-unit, a *decibel,* is usually used. For L in decibels, the formula is as follows: $L = 10 \cdot \log \frac{I}{I_0}$

A sound level meter is the principal instrument for general noise measurement. The indication on a sound level meter indicates the sound pressure, p, as a level referenced to 0,00002 Pa, calibrated on a decibel scale.

Sound Pressure Level = $20 \cdot \log \frac{p}{0,00002} dB$

We can also search for some explanations on the Internet:

http://online.anu.edu.au/ITA/ACAT/drw/PPofM/intensity/Intensity1.html
http://www.point-and-click.com/Campanella_Acoustics/faq/faq.htm#basic_decibel

Based on these explanations, students have to answer various questions about the noise in our environment. So they have to solve various examples of logarithmic equations and not surprisingly, even those who always make remarks about how mathematics is boring, try to find out how much more intensity there is in a rock concert than in whispering to the ear of his/her friend. They also discover if we can simply add up intensity of noise. The worksheet can be found on

http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/sound.htm

Dimensions of Fractals

In this last example we enter a field of more modern mathematics. Students often hear about fractals in art and design. So it is almost necessary to introduce the mathematical view of this topic as well. The before-mentioned examples can also be also solved with the aid of "non-symbolic" calculators. However, with fractals the aid of a computer is crucial. Students draw a few iterations of fractals and see how they develop. This helps them when calculating their dimension.

The dimension of a fractal is very interesting. We are used to the idea that a line is one-dimensional, a plane two-dimensional, and a solid three-dimensional. But in the world of fractals, dimension acquires a broader meaning, and need not be a whole number. We only study just some simple examples, such as the dimension of the Sierpinski gasket. Fractals with their rational dimension can be an introduction to the new topic; geometry of 2D and 3D space.

Students first encounter with fractals is not in the classroom. As their homework they have to find out very thing they can about fractals. In the class we take a look at the collected data, show some interesting pictures of fractals – from the simple ones to the artistic ones and some fractals from the nature as well (trees, cauliflower, broccoli, …). After this kind of introduction the students are prepared for solving the exercise.
The worksheet is on http://rc.fmf.uni-lj.si/matija/logarithm/worksheets/fractal.htm.
Several students showed so much interest in this subject that they agreed to prepare their own "research work" in the next school year. Again we provide them with some WWW links as starting points.

- Catalogue of fractals: http://sprott.physics.wisc.edu/fractals.htm
- All on fractals: Koch snowflake, self-similarity, dimensions http://math.rice.edu/~lanius/frac/. Suitable for beginners also.
- A Journey into Menger's sponge, http://pages.hotbot.com/arts/werbeck/return.html

- More detailed and concise explanations can also be found on
    - http://mathworld.wolfram.com/Fractal.html
    - http://www.ncsa.uiuc.edu/Edu/Fractal/Fractal_Home.html
    - http://www.ics.uci.edu/~eppstein/junkyard/fractal.html
    - http://forum.swarthmore.edu/alejandre/applet.mandelbrot.html
    - http://www.swin.edu.au/astronomy/pbourke/fractals/fracintro

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