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ALGEBRAIC INSIGHT AND STUDENTS' USE OF DERIVE

Robyn Pierce
University of Ballarat, Australia
<r.pierce@ballarat.edu.au>

Kaye Stacey
University of Melbourne, Australia
<k.stacey@edfac.unimelb.edu.au>

Introduction

The use of Computer Algebra Systems (CAS), like DERIVE, for teaching and learning mathematics has raised many questions. This paper reports on a small-scale study that focused on two questions: firstly, does level of algebraic insight relate to students' use of CAS? Secondly, does algebraic insight improve in a mathematics course taught using CAS? The paper first discusses the concept of algebraic insight, analysing it in terms of three key components. Several aspects of students' use of CAS are then described. The results of the study outline features of the students' use of CAS and their algebraic insight and reports on a preliminary exploration of the links between them.

Background

Algebraic Insight

Algebraic insight is a broad, complex and inexact concept that develops as a student learns mathematics. It requires the mathematician to develop a structural view of symbolic expressions. Algebraic insight may be seen as made up of three key components that we call (a) algebraic estimation, (b) linking between representations and (c) good choices and interpretations of symbols. These key elements will always be part of algebraic insight but evidence of facility that should be expected with these elements will be context dependent. In this study examples focus on aspects which are particularly relevant to users of computer algebra systems who are studying functions and introductory calculus. In the following section, the key elements are described.

Algebraic estimation

Algebraic estimation requires a combination of conceptual understandings and operational skills. It involves recognising structures and patterns in expressions and thus either predicting the form of solutions or being alert to errors and impossibilities. Students with good algebraic estimation skills will be able to detect equivalent expressions, recognising whether or not one expression may be manipulated to produce a second matching alternative. This skill is part of what Fey (1990) and Arcarvi (1994) term 'symbol sense'.

We have chosen the name algebraic estimation because this is the algebraic equivalent of arithmetic estimation skills – where a quick assessment of the expected characteristics of an expression is made. For example arithmetic estimation may involve expecting an answer in the millions; algebraic estimation may involve expecting a polynomial of order six. Algebraic estimation relies on three sub-components listed below.

Awareness of conventions: This awareness means knowing, for example, that a multiplication sign is implicit between two adjacent symbols, e.g. that 2*ab* means 2**a***b*, in conventional mathematics. It is important to be aware of this convention when entering or interpreting expressions using a computer algebra system which allow the user to define variables with 'names' of more than one letter. So, for example, a variable named "ab" must be distinguished from a*b, the product of variables "a" and "b" Similarly it is important for students to be aware that the order of operations for a mathematical expression follows an agreed hierarchy of operations not a left to right sequence. When using a computer algebra system this will require careful bracketing of expressions.

Awareness of structure and identifying key features*:* Algebraic fractions, for example, require 'seeing' an expression in its parts, noticing detail and realising that detail matters. In particular $\frac{a}{c} + \frac{b}{d}$ needs to be viewed as two fractions $\left[\frac{a}{c}\right] + \left[\frac{b}{d}\right]$ or one fraction $\frac{ad + bc}{cd}$ as opposed to $\frac{a+b}{c+d}$ which must be seen as one fraction $\frac{[a+b]}{[c+d]}$. Recognising what part of an expression should form a numerator and what a denominator is important for entering expressions into a CAS and copying results from a screen. To detect equivalent expressions it is important to notice the detail in expressions that appear similar but are different.

Identifying key features is part of a structural, global view of algebra which is described by many writers on algebra education (for example Sfard, 1991, Thomas and Tall, 1991, Kieran, 1992) and distinguished from a procedural view. The power of a polynomial, for example, is one of the first and simplest features to note when checking for equivalent expressions. It is also a key indicator for the shape of a graph. It indicates the family of functions to which an equation belongs. This is another feature of symbols sense which Fey (1990) suggested was important for users of CAS to acquire.

Using and recognising composite functions requires a structural view of algebra. Seeing the pattern of imbedded objects allows students to find the common factor in Thomas and Tall's (1991) $(2x+1)^2 - 3x(2x+1)$ example or in later trigonometric functions to follow that an expression like $\sin^2 x + 2\sin x \cos x + \cos^2 x$ may equally be written as $(\sin x + \cos x)^2$ or $\sin 2x + 1$. Algebraic insight is not so much shown in the detail but in being able to say '(2*x*+1) is common to both terms' or 'This follows the pattern for a perfect square' or '$\sin^2 x + \cos^2 x = 1$'.

This global view also allows students to make the series of substitutions necessary, for example, to find $f(g(x))$ if $g(x) = x+h$ and $f(x) = 5x^3 - 2x^2 + x + 7$. To show algebraic insight a student would not need to perform the expansion of this new expression but realise that every *x* must be replaced with the new *(x+h).*

Linking between representations

Fey (1990) noted that an understanding of algebra would be demonstrated by students' ability to move between algebraic, numeric (tabular) and graphical representations of expressions. This is the second component of algebraic insight. Students would demonstrate algebraic insight by being able to have a rough idea what a graph of a function may or may not look like, have some idea of the sort of range of values that might be expected for a given domain. To show algebraic insight for linking between representations, a student would, for example, not be expected to complete a least squares linear regression but to recognise that a set of data points might be modelled by a linear function. They would be looking for a function or class of functions to link between the dependent and independent variables. MacGregor and Stacey (1993) found that this was not the way many students viewed tabular data tending instead to look at individual values or patterns in the dependent variable alone. . Similarly a student would immediately recognise a quadratic polynomial as being represented by a parabolic graph and, as Arcavi (1994) suggests, the student may realise that each representation is equally valid and that changing representations may help them make progress in solving a problem.

Good choices and interpretations of symbols

Many writers (for example, MacGregor and Stacey, 1997) remind us that students often view letters and symbols in different and more limited ways than we expect. To demonstrate algebraic insight students need to be able to appropriately use and interpret letters. Booth (1990) discussing students understanding of algebra and Arcavi (1994) describing attributes of symbol sense both emphasise that students need to be able to interpret a letter as standing for a single unknown, general unknown or variable depending on the context of the problem because symbols may play different roles in different problems. Arcarvi (1994) points out that symbol sense is not only shown in selecting an appropriate symbolic representation of a problem but also in recognising when symbolic representation is not the best starting point. It is also shown in interpreting symbolic solutions in the context of problems.

Students' use of CAS

The second area of interest in this study is to map and describe students' use of CAS. In considering this, we must look at any barriers that the technology presents in the way of technical difficulties and other practical problems. Next the various manners in which students may approach using CAS are noted and the purposes for which students choose to use CAS are considered.

Technical difficulties and practical problems

For students to use a CAS effectively it must present them with few technical difficulties. Hunter, Marshall, Monaghan and Roper (1995) warned that technical difficulties can create cognitive noise for students. Firstly, the overhead cost involved in learning the appropriate syntax and command language should not distract from the mathematics. Concern has been expressed, by Atkins, Creegan and Soan (1995), that students trying to learn new mathematics combined with new syntax may lose sight of the objective, especially when long sequences of commands are required.

Secondly, the interpretation of the results produced by the CAS may not be as evident to the novice as to the expert mathematician. Setting an appropriate scale for a graph window and sketching the graph with its key features is a skill that must be developed not just assumed. Examples of students' difficulties with using and interpreting CAS graphs are given by Hillel and Linchevski (1992), Schoenfeld (1992), Hunter, Marshal, Monaghan and Roper (1995), and Pierce (1999).

The manner in which students choose to use CAS

Students approach the use of CAS in different ways. Arnold identified five methods of using such software. He described these as levels along a continuum from those who prefer not to use CAS to those who choose to use CAS on efficient and effective ways.

> Level 0: *Non-Use:* Although the software is available and appropriate, and the user has sufficient skill to use it, no use is made.
> Level 1: *Passive*: The user is content for the tools to be operated by another, but takes no personal initiative.
> Level 2: *Random*: Use is not goal-directed and bears no relation to the context.
> Level 3: *Reflexive*: The user makes superficial and automatic use of appropriate tools.
> Level 4: *Strategic*: Use of the tools is deliberate, goal-directed and insightful. (Arnold, 1995, pp. 321-322)

The purpose for which CAS is used

The purpose of the use may be primarily functional (to get answers) or pedagogical (to learn from). Functionally, students will use CAS to find answers to problems they could have 'easily' have done with pen and paper however its main use should be in enabling students to find answers to 'hard' questions or time consuming questions. Studies such as Bennet (1995) report that students find that using CAS helps them avoid making 'silly mistakes'.

CAS may be used for pedagogical purposes. CAS may be used, for example, to repeat a process a number of times in order to establish a pattern. The mathematics may not be difficult to do by hand but may be time consuming and boring. Students may use CAS to explore alternative and, sometimes, unconventional methods of solution. (for examples see Bennett, (1995) or Atkins, Creegan, and Soan, (1995)). In addition students may explore variations on problems, looking to see 'what happens if…?' In these ways CAS can be used to develop inductive reasoning (Mayes, 1995) and engage students in their learning of mathematics (Heid, 1995).

This study

As the place of CAS in teaching and learning mathematics is considered, it is worth exploring the links between Algebraic Insight and the use of CAS. Do students with poor Algebraic Insight have more difficulty using the program? . How do students of different levels of insight vary in their use of CAS? How do they vary in the benefits, if any, they gain from a mathematics course taught with CAS available?

The research exploring these questions was carried out during the second semester of 1999 with a group of first year undergraduates studying an introductory calculus course. The researcher, an experienced teacher, taught this course. Twenty students enrolled in the course but only fifteen completed all assessable tasks. These students were enrolled in a variety of degree programs but most were training to be primary or secondary teachers.

Data sources

Data was gathered throughout the semester, using the data sources described below, along with the number of responses obtained for each administration of each instrument.

Background survey (n=20)

The purpose of this questionnaire was to establish students' highest level of school mathematics and their previous experience in using technology for mathematics.

Algebraic Insight Quiz: Pre-course (n=20), Mid-course (n=12), Post-course (n=15)

This multiple-choice quiz was presented to the students as a 'Powerpoint' slide show. The questions were each displayed for between 5 and 15 seconds with students indicating their responses on an answer sheet provided. The purpose of this quiz was to assess two of the elements of algebraic insight: students' basic algebraic estimation skills and their ability to link numeric, graphical and algebraic representations of functions quickly. A different instrument (not reported in this paper) was used to assess good choices and interpretations of symbols.

Lagrange (1999) agrees that for students to use CAS effectively they need to be able to recognise equivalent expressions. The quiz aimed to put this concept of equivalence into a context with which students would be familiar, that is checking solutions to questions against text book solutions. In the first 11 items (like items 5 and 10, figure 1) students were presented with pairs of expressions and asked to compare them as if they were checking answers. They were asked to decide if the 'student' was definitely right , possibly right, definitely wrong or possibly wrong.

The options 'possibly right' and 'possibly wrong' were included, since what was required was only an intuitive response. To discourage from randomly guessing, the option 'I have no idea' was offered as a response. The later, multiple choice questions (like items 17 and 21 in figure 1), assessed students' ability to translate between representations of functions.

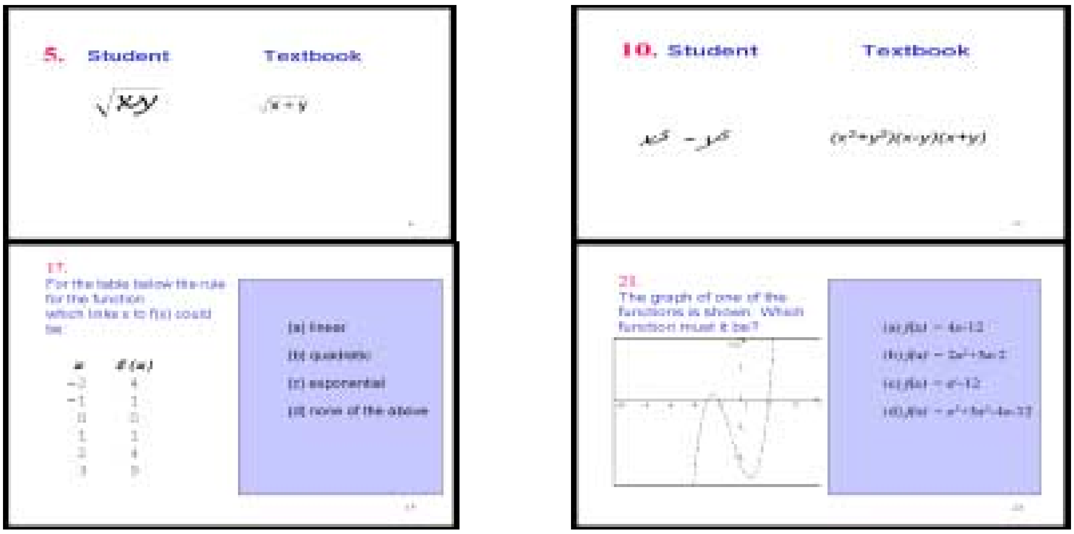

Figure 1: Examples of Algebraic Insight Quiz items

Mathematics staff and volunteer students, from other courses, tested the Algebraic Insight Quiz. Items were revised as a result of their feedback. The quiz proved to be a quick and reasonably valid method of assessing the students' algebraic insight. There were significant (α=0.05) positive correlations between the pre-course quiz, mid-course quiz, and post course examination marks (r = 0.66 and r = 0.56) and between the pre-course quiz and students' overall course marks (r = 0.73). These results support the reliability of the quiz.

Unfortunately circumstances beyond the control of the researcher affected some student's performance on the post-course quiz. To compensate for this students' maximum scores for mid and post-course quizzes were recorded as 'later quiz' results

CAS use survey: Early-course (n=15), Mid-course(n=13), and Post-course(n=12)

This survey monitored students' use of DERIVE including their difficulties in using the program and the purposes for which they used CAS. In the first section students were asked to circle the word which best indicated the number of problems they had during that day's computer laboratory session with any of the listed tasks (examples, figure 2).

	Not Applicable	**None**	**One**	**Some**	**A lot**	**Every time**
Authoring , using /, ^ , *, +, - symbols	**NA**	**N**	**O**	**S**	**A**	**E**
Interpreting the results of the soLve command	**NA**	**N**	**O**	**S**	**A**	**E**
Working from the screen to ordinary maths symbols	**NA**	**N**	**O**	**S**	**A**	**E**
Working out the scale of a graph	**NA**	**N**	**O**	**S**	**A**	**E**

Figure 2: Examples of items on CAS-Use survey monitoring technical difficulties and practical problems.

In the second section of the survey students were asked to reflect on reasons for their use of DERIVE and to tick those statements, shown in figure 3, which applied to them.

Today I have used DERIVE to:	
Find answers if the computer was suggested.	[ ]
Explore problems if the computer was suggested	[ ]
Explore variations on the set problems.	[ ]
Explore, other than when directed but on the same topic	[ ]
Explore other mathematics not on today's topic	[ ]
Find answers I could 'easily' have done with pen and paper.	[ ]
Find answers to 'hard' questions	[ ]
Find answers to time consuming questions.	[ ]

Figure 3: Examples of items on CAS use survey monitoring the manner and purpose of CAS USE

Course Overview (n=13)

This sought to establish whether students found DERIVE helpful for their learning and if so in what way. These items, shown in figure 4, linked students' use of CAS and learning of mathematics. Students were asked to respond using a five-point Likert scale from strongly disagree to strongly agree. Items one to five focus on the purpose of CAS use and items six and seven on the manner of CAS use while items eight to eleven ask the student to reflect on the learning outcomes of this computer use.

1. Derive has helped me see patterns in Mathematics.
2. I find using Derive helps me to understand Mathematics
3. DERIVE can be made to do the working out for maths problems.
4. I can use DERIVE to check every step of a problem.
5. A DERIVE plot will tell me everything I need to know about a function.
6. I only use DERIVE if the instructions tell me to.
7. I try out ideas using Derive.
8. Compared to my confidence with *functions* before this unit, my confidence now is much greater.
9. Compared to my confidence with *calculus* before this unit, my confidence now is much greater.
10. Compared to my confidence with *graphs* before this unit, my confidence now is much greater.
11. Compared to my confidence with *trigonometry* before this unit, my confidence now is much greater.

Figure 4: Items linking manner and purpose of CAS use and learning outcomes

Mid-course examination (n=18), Post-course examination (n=15)

The examinations assessed the students' understanding of the course material. Questions required more detailed solutions than that obtained by simply using a command like "differentiate". Overall course marks combined assessment from these two examinations with two assignments.

Observations

Observations of students' use of CAS, during both laboratory classes and examinations were made by the teacher/researcher and notes were made immediately following each session.

Results & discussion

Preliminary analysis of the data shows the following:

Algebraic Insight

Pre-course algebraic insight quiz results were poor with the mean score being only 51%. This result was not surprising since most of the students taking this course had studied little algebra in their final year of secondary school. In addition it is to be noted that 80% of this University's students fit at least one of the government's 'disadvantaged' categories for the purpose of tertiary education. It was pleasing that later testing showed an improvement to a mean score of 66%.

Over the period of the course, as a group, students showed some improvement in recognising and applying conventions, identifying structure and key features of expressions and linking representations (Table 1). Within the items relating to awareness of structure and key features students showed improvement in identifying key features like the highest power of the polynomial but still had difficulty recognising structure in algebraic fractions even though this was targeting in their worksheets. Improvement in students' results on items testing their ability to link representations was limited because this section showed the best results in the first quiz. There were moderate, positive, correlations of the three quiz results suggesting not all students responded in the same way to the teaching. The variability of the group's results was reduced (pre-course: $s^2 = 17.98$ later-course: $s^2 = 7.56$). The weaker students improved while the highest scorers did not. For the very best students this quiz gave little opportunity to show improvement during the course.

Table 1: Algebraic Insight Quiz results

Quiz Section	**Number of items**	**Pre-course mean**	**Later mean**	**t-statistic**	**df**	**significance (two-tail)**
Awareness of conventions	4	2.6	3.2	2.71	9	0.024
Awareness of structure and identifying key features	11	4.7	6.5	2.21	9	0.054
Linking between representations	7	4.5	5.5	2.12	9	0.062
Total quiz	23	11.8	15.2	2.89	9	0.018

Students use of CAS

On the background survey students were asked to rate their 'general computing skills' on a scale from 0 to 10 where 1 was labelled poor and 10 excellent. When these responses were plotted against a measure of number of their difficulties in using CAS reported on the Early-course survey a negative tend was evident but the correlation was not significant. The student with the highest general computing skills reported the fewest difficulties in using DERIVE but the next group who reported few difficulties considered their computing skills were quite poor. The researchers observations suggest that these students were less confident in their use of CAS but took more care to follow instructions, so made less errors. Initially they used CAS in a passive, then reflexive manner. This means that they moved from Arnold's level 2 to level 4 of manner of CAS use.

Students' technical difficulties in using CAS were classified into three categories: inputting, interpreting and graphing. All students reported fewer problems at mid-course than they had at the beginning but more problems later in the course. Observation by the researcher suggests that by the end of the course students were using the CAS to do less familiar mathematics and were showing more initiative, trying to be strategic, in their use of CAS. At the end of the course only one student agreed that they 'only used DERIVE if the instructions told me to'. This wider use of CAS may account for the increased number of errors.

It was expected that students with poor algebraic insight would have the most difficulties, initially, with the 'Inputting' component of CAS use, but the results do not support this. Students were grouped on the basis of their pre-course algebraic insight quiz (poor < 40% and OK otherwise) and early course Inputting difficulties (few and lots). A Fisher's exact test supported the independence of these variables ($p = 1.000$).

Links between Algebraic Insight and CAS

Students' responses to the course overview survey suggest that using CAS in a strategic manner for pedagogical purposes had positive learning outcomes. it . There was a significant positive correlation ($r = .76$) between the responses to 'I try out ideas using DERIVE' and 'DERIVE has helped me to see patterns in mathematics'. Those students using DERIVE for pedagogical purposes were experiencing positive learning outcomes. It was interesting to note that both the student who scored lowest and the two highest scoring students on the Algebraic Insight Quiz reported agreeing with the statement 'I try out ideas with DERIVE'. The lowest and one high scoring student strongly agreed that 'using DERIVE helps me understand mathematics', however the second high scoring student disagreed. This student reported her general computing skills as good and had few difficulties using the program but repeatedly commented that she felt "scared of the computers". Her preference would have been to be a passive or non-user of CAS. A further interesting result emerged when the data was looked at a second time, excluding the top score for Algebraic Insight. A correlation of Algebraic Insight scores from the pre-course quiz with students' responses to 'I find using DERIVE helps me to understand mathematics' yielded a significant negative result ($r = -0.80$). Weaker students responded more positively to the use of CAS. This trend is illustrated by figure 5.

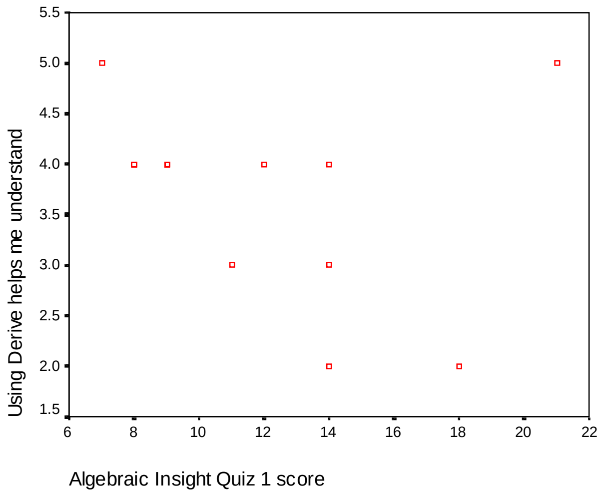

Figure 5: 'Using DERIVE helps me understand maths' against Algebraic Insight Score

On the course overview students were asked to compare their level of confidence in different sections of mathematics after this course to that before studying this course. Those who said that they tried out ideas using DERIVE were more likely to report improved confidence with graphs and functions. Those agreeing with the statements 'I find using DERIVE helps me understand mathematics' were more likely to report improved confidence with graphs. Most students, (8 out of 9) who responded positively to this statement also showed improvement in Algebraic Insight (Pre-course versus Later-course). The ninth student scored highest on the first quiz and had little room for improvement. Three students indicated neutral or negative response to the statement and only one of these students showed improvement in algebraic insight.

Conclusions

Preliminary analysis of the data suggests that students' responses to using DERIVE for learning mathematics were highly individual. Students with good Algebraic Insight and good general computing skills did not necessarily have fewer difficulties with using DERIVE than those with weak backgrounds. Motivation and concentration seemed to play as big a part as background skills.

Assessing Algebraic Insight is a difficult task and the quiz used in this part of the study addressed only Estimation and Linking between representations. The quiz allowed the researcher to roughly order students and highlighted areas of strength and weakness but would be improved by the addition of more items in each section.

Overall students did show improvement in Algebraic Insight but this was neither as strong nor as consistent as hoped. It did, however, appear that those who moved past just using DERIVE when instructed to find answers (that is functional, reflexive use as in Arnold's level 4) to trying ideas out using DERIVE, (that is strategic, pedagogical use as in Arnold's level 5), were more likely to show improvement in Algebraic Insight and mathematical confidence. The results also suggested that, on the whole, students with poor Algebraic Insight gained more benefit from the use of DERIVE than stronger students.

Future papers will report the above results more thoroughly and describe the concept of algebraic insight more closely.

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