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The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics

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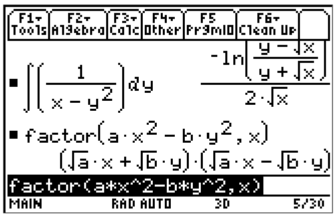
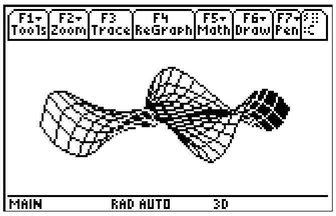
Introduction

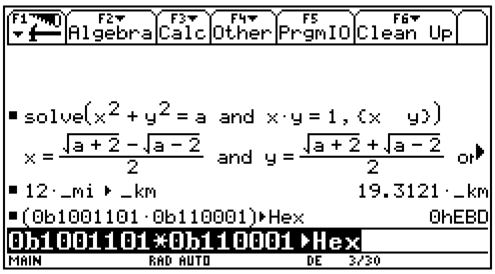
a) Algebraic Calculator

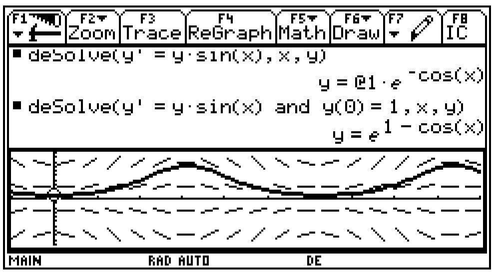
Algebraic Calculators are graphic calculators, which offer features that so far were only available on computers. These calculators can simplify expressions, differentiate, integrate, and plot functions, solve equations, manipulate matrices, etc. In short: They can do most of what we teach in mathematics at schools and colleges.

Following are two screen images each of the algebraic calculators TI-89 and the TI-92. The first picture demonstrates the calculation of an antiderivative (indefinite integral) and the decomposition of a polynomial into linear factors. The second picture shows a 3-dimensional graph, which even can be rotated in real-time. The third picture shows the solution of a system of non-linear equations, a physical units conversion, and the conversion of the result of a binary number calculation into the hexadecimal system. The fourth picture provides you with the algebraic and the numeric/graphic solution of a differential equation.









Such calculators soon will be the standard tool as is the scientific or graphic calculator today. In this article I discuss in detail how this technology is likely to effect the teaching of mathematics. As an introduction I look at the role of technology in general.

b) Technology

The word 'technology' comes from the Greek word *technikos* which means "artistic, expert, professional". In the form of computers, technology enters more and more areas of life. In this article

I look at two areas and explain the importance and significance of technology therein. The two areas are *mathematics* (intellectual) and *moving/ transportation* (physical). This comparison was stimulated by Frank Demana.

The most elementary method of moving is *walking*. Walking is a physical achievement obtained with mere muscle power. The corresponding activity in mathematics is *mental calculation* (mental arithmetic and mental algebra.) Mental calculation requires nothing but “brain power”.

Riding a bicycle is a method of moving, where we employ a mechanical device for making more effective use of our muscle power. Compared to walking we can move greater distances or faster. The corresponding activity in mathematics is *paper and pencil calculation*. We use paper and pencil as “external memory” which allows us to use our brainpower more efficiently.

Another method of moving is *driving a car*. The car is a device that produces movement. The driver needs (almost) no muscle power for driving, but needs new skills: He must be able to start the engine, to accelerate, to steer, to brake, to stick to the traffic regulations, etc. The corresponding activity in mathematics is *calculator/ computer calculation*. The calculator or computer produces the result, while its user needs to know how to operate it.

What method of moving is sensible in which situation? If we ask a colleague to get a newspaper from a 250-meter distant newsstand, he probably will walk. In case the newsstand is 1,000 meters away, a bicycle may be the most reasonable means of transportation. In case the distance to the shop is 10,000 meters, one will use a car. In mathematics, the sensible use of technology is accordingly: The multiplication of two one-digit numbers is best done mentally. Two two-digit numbers can well be multiplied using paper and pencil, while for the product of two five-digit numbers one will use a calculator.

One could throw in that “*many students use a calculator to obtain the product of 7 and 9*”, hence they are likely to loose the skill of performing mental arithmetic. This is a clear case of improper use of technology, which happens not just in mathematics. Some people misuse their car by driving 250 meters to the next newsstand. Those, who do so, harm themselves (lack of physical exercise) and our environment (through the exhaust fumes). Despite the possible misuse of the car we do not demand its abolition. Similarly we should not banish calculators and computers just because some students might use them improperly. As much as we needed (and still need) to create a general awareness that physical exercises are essential for physical fitness and health, we need to create the awareness that intellectual exercises (mental arithmetic and mental algebra are such exercises) are essential for intellectual fitness and health. More on this will be discussed in the last but first section.

The analogy is not finished yet. What, if the colleague, who we asked to get a newspaper from a 250 meter away newsstand, can’t walk properly (because he is physically challenged or has a broken leg)? For him, walking 250 meters can be very difficult, if not undoable. There is technology available to help these people, for example a wheel chair. *Using a wheel chair* is a method of technology-supported moving, where a physical weakness is compensated. In an intellectual activity weaknesses could occur as well, whose compensation is desirable if not necessary. I give an example from mathematics teaching: A student with a weakness in solving systems of linear equations will find it difficult to solve analytic geometry problems, simply because the solving of a system of equations is a subproblem encountered frequently in analytic geometry. It is not only an act of humanity, but our pedagogical duty to provide this student with a tool which compensates his or her weakness, hence allowing the student to do analytic geometry properly despite the weakness.

As we will demonstrate later with more examples, calculators and computers can be excellent mathematical compensation tools which allow less gifted students to deal with advanced topics. It goes without saying, that the ultimate goal in mathematics teaching is to weed out all weaknesses in skills that are regarded essential. A physically challenged person need not be tied to a wheel chair for the rest of his life. A physician will endeavour to repair a patient's physical disability as much as possible, using an individual therapy. Similarly, a teacher should endeavour to repair a student's intellectual/mathematical disability with a proper, individual therapy. In both cases we will facilitate the patient's "daily life with the disability" – i.e. the time outside the therapy – by providing an appropriate compensation tool (wheel chair, calculator).

Following is a summary of the analogy.

Moving/Transportation	**Mathematics**
physical	*intellectual*
walking	mental calculation
riding a bicycle	paper & pencil calculation
driving a car	calculator/computer calculation (automation)
using a wheel chair	calculator/computer calculation (compensation)

Teaching with Technology

Based on what we explained in the previous section, we distinguish two elementary uses of calculators or computers in teaching: automation and compensation. Based on these two application types, we demonstrate how one can use calculators and computers as valuable teaching tools by looking at four topics which I think are especially important in mathematics education: trivialisation, experimentation, visualisation, and concentration.

a) Trivialisation

The car broadened our "moving/transportation horizons" by trivialising moving up to certain distances. Similarly, the calculator broadens our "calculation horizons".

Remember the "old days" before scientific calculators? Exam questions or homework problems had to be chosen very carefully so that all intermediate and final results were "nice". A "nice" result was an integer, a simple fraction, or a simple radical which, later in the calculation, often would disappear again. This was important, as otherwise the students would have had to use most of their time performing arithmetic operations. With a scientific calculator one can multiply two seven-digit numbers as quickly as two one-digit numbers. The scientific calculator trivialises the performing of arithmetic operations.

Drawing the graph of a linear function (e.g. $y=2x+3$) is simple once you know the geometric meaning of the two coefficients. Only a glimpse of talent and a ruler are enough to produce a proper graph.

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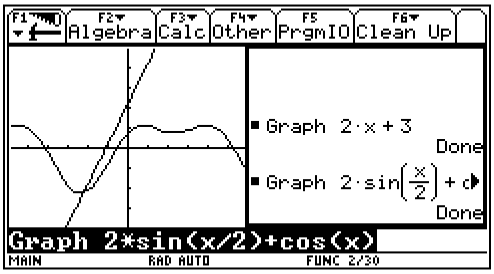
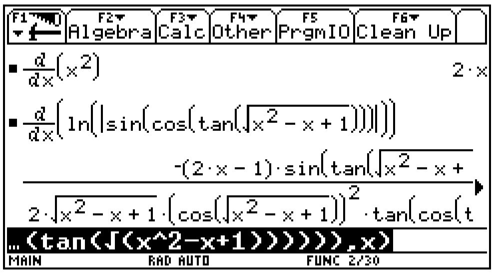
Drawing the graph of a function such as y=2sin(x/2)+cos(x) is much more difficult and the production of a proper graph requires a reasonable degree of talent for drawing. With a graphics calculator one can plot both functions within the same amount of time and talent. The graphics calculator trivialises the production of graphs.



Computing the first derivative of y=x^2 is simple once you know the differentiation rule for powers. Determining the first derivative of $y=\ln(\left|\sin(\cos(\tan(\sqrt{x^2-x+1})))\right|)$, however, is a lot of work even for a good mathematician. The algebraic calculator can manage both examples within seconds. The algebraic calculator trivialises algebraic (symbolic) computations. A landmark paper about



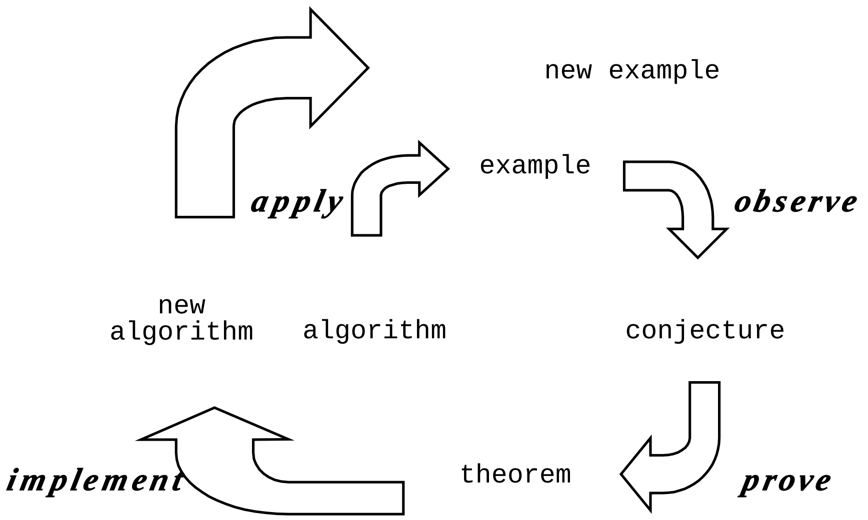
trivialisation of algebraic computation in mathematics teaching is [Buchberger 1989], where the WhiteBox/BlackBox principle is introduced.

The car (i.e. the trivialisation of moving and transportation) made a distance such as Los Angeles to San Diego easily manageable. In other words, moving and transportation tasks that have been considered difficult in earlier times nowadays are fulfilled routinely with cars. Similarly, calculators and computers (i.e. the trivialisation of arithmetic, graphics, and algebra) in teaching mean that we can tackle

- (more) complex problems and
- (more) realistic problems.

b) Experimentation

How did we discover all the mathematical knowledge we know today and how do we find more mathematical knowledge? According to one of the epistemologically oriented theories one can visualise the main steps of these discoveries as follows: Applying known algorithms produces *examples*. From the examples we *observe* properties, which are expressed as a *conjecture*. Proving the conjecture yields a *theorem*, i.e. guaranteed knowledge. The theorem's algorithmically usable knowledge is *implemented* in a new *algorithm*. Then the algorithm is *applied* to new data, yielding *new examples*, which lead to new observations, ...



This picture of a spiral, which demonstrates the path of discovery of (mathematical) knowledge, was proposed by Bruno Buchberger. A detailed description of *Buchberger's Creativity Spiral* and references to related models can be found in the highly recommended (German language) book [Heugl/Klinger/Lechner 1996].

In this spiral we find three phases. During the *phase of experimentation* one applies known algorithms to generate examples, then obtains conjectures through observation. During the *phase of exactification* conjectures are turned into theorems through the method of proving, then algorithmically useful knowledge is implemented as algorithms. During the *phase of application* one applies algorithms to real or fictitious data. Typically, the solution of real problems serves the purpose of mastering or facilitating life, while the solution of fictitious problems serves the purpose of entertainment/diversion (e.g. mind puzzles) or the finding of new knowledge (i.e. the satisfaction of scientific curiosity).

Mathematics is about 5,000 years old. The first 2,500 years it was an experimental science and belonged to the cultural assets of the Egyptians and other ancient civilisations. Using the above notions, it consisted only of the phases of experimentation and application. About 500 B.C. the Greek took the Egyptian mathematics and applied to it the deductive methods of their philosophy (i.e. they added the phase of exactification), thus establishing mathematics as the deductive science as we know it today. From then on scientific mathematics comprised all three phases. After the mathematical knowledge grew for a while, a group around the French mathematician Dieudonne (the group became known under the name “Bourbaki”) undertook a project aiming at writing down the mathematical knowledge of their time in a uniform and concise manner. They developed the system of “definition-theorem-proof-corollary-...”, which has become characteristic to modern mathematics. This Bourbaki system, being developed for the purpose of inner-mathematical communication, did without documenting the phase of experimentation, hence it consisted only of the phases of exactification and application.

Gradually, Bourbakism lodged itself in teaching and learning. It has become customary to teach mathematics by deductively presenting mathematical knowledge, then asking the students to learn it and use/apply it to solve home work and exam problems. This is as if one would have to learn walking (or cycling, or dancing, ...) by studying, understanding, then applying scientific descriptions of the muscle motions required for walking (or cycling, or dancing, ...) – instead of learning by trial and error (i.e. experimentation), as it is done naturally. Most of today’s psychological theories of learning consider learning to be an inductive process in which experimentation plays an important role. This is why Freudenthal demanded that we should not teach students something that they could discover themselves (see [Freudenthal 1979]).

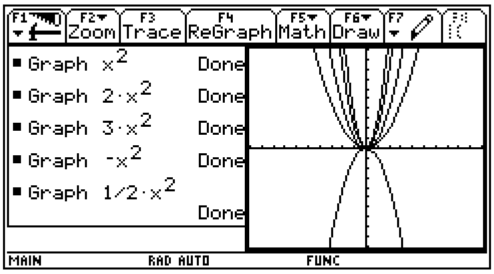
Hardly any mathematician on this planet could do mathematical research the way we demand our students to get into this subject. A student has to “locally” build his individual little “house of mathematics”. A scientist does pretty much the same “globally”, i.e. on a much larger scale. For both the scientist and the student a substantial part of knowledge acquisition happens during the phase of experimentation. From this point of view it becomes understandable why so many students are at loggerheads with mathematics and one will demand that experimentation obtains its due position within the teaching of mathematics. Phases of experimentation should complete the traditional teaching methods – not substitute them! We do not advocate the return to Egyptian experimental mathematics. Simply, mathematics teaching should go through all three phases of the above spiral.

However, it is understandable that, within the framework of today‘s curricula, there was hardly any experimentation in the sense of Buchberger’s creativity spiral. This kind of experimentation, performed with paper and pencil, is both time-consuming and error prone. Within the time available at school, students could produce only a very small number of examples for the purpose of observing and discovering, and a hefty portion of these examples could be faulty due to calculation errors. There is nothing you can observe from only a few, partly wrong examples! From now on algebraic calculators enable students to experiment within almost all topics treated in mathematics teaching. There is no limit to the number of examples the student can do and the electronic assistant guarantees the correctness of the results. Talking about an assistant: Records indicate that great mathematicians such as Carl Friedrich Gauss employed herds of human “calculators” without which they would not have made most of their famous findings.

The great genius Johann Wolfgang Goethe called for “learning through doing and observing”. Using algebraic calculators we can now meet Goethe’s demand.

c) Visualisation

Visualisation means illustration of an object, fact, or process. The result can be graphic, numeric, or algebraic. Today, the term is mostly used for graphic illustrations of algebraic or numeric objects or facts. (The term is used either for the process of illustration or for the result of the illustration process.) Visualisation as a technique of teaching mathematics has become important mostly in those countries, where graphics calculators are widely used. Today it is mainly used to acquire the competence of changing between representations, mostly for the purpose of studying the correspondence between algebraic and graphic representations. A typical example is studying how the parameter *a* effects the shape of the graph of the function $y = a \cdot x^2$:



Frank Demana and Bert Waits are the leading advocates of a teaching style that is called the “power of visualisation” (see [Demana/Waits 1990, 1992, 1994]).

In the psychology of learning scientists discovered the concept of *reinforcement* and showed, that reinforcement works best if it follows the action immediately. An example from everyday life is a child who puts its hand on a hot stove. The immediate pain is the best prerequisite for the child to learn not to do this anymore. If the pain would be felt only several minutes later, the child probably would not connect it with the (long ago) touching of the hotplate and it probably would not learn anything.

When using a calculator as a visualisation tool, the immediate feedback is of central importance. If you enter into your calculator, for example, x^2, then press the right key, the corresponding graph appears only fractions of a second later. The resulting picture can be discussed, the graph can be associated with the expression, etc. Consider a less gifted student and compare the learning effect when the student has to draw the graph manually with when the student is allowed to use a calculator: A manually produced graph would take too much time and most probably it would have only a vague resemblance with the true graph. (“What can you observe from wrong examples?”) Only with the help of the calculator this student has a realistic chance to memorise the correspondence between the expression and the (proper!) graph. Producing graphs with paper and pencil certainly continues to be a worthwhile activity which is important in learning to *understand* the correspondence between algebraic and graphic representations. However, immediacy and correctness are such crucial psychological factors, that providing less gifted students with an appropriate tool (such as a graphic calculator) is a pedagogical duty.

If and how a teacher uses a tool for supporting an activity depends on the pedagogical goals connected with the activity. Helmut Heugl once said *“If it is not (pedagogically) necessary to use an algebraic calculator, it is (pedagogically) necessary not to use the algebraic calculator.”* This means, in particular, that the teacher becomes more and more important in a technology-supported mathematics education, hence the importance of teacher preservice and inservice training grows.

d) Concentration

We can compare teaching and learning mathematics with building a house, the “house of mathematics”. The topics which we teach and the dependencies between them are comparable to the storeys of a house. Before one can build the house’s second storey, one has to complete its first.

Similarly, the treatment of almost any mathematical topic requires the mastery of earlier learned topics. We demonstrate this using the topic “solving of a linear equation in one variable”.

We look at the equation $5x\text{-}6=2x+15$. One has to transform it into the form x=... . This is achieved through choosing and applying an appropriate sequence of equivalence transformations. Typically, the student is advised to “bring terms with x to one side of the equation” and to “bring all other terms to the other side”. Therefore we start by subtracting $2x$:

$$5x-6 = 2x+15 \qquad | \; -2x$$

After *choosing* this equivalence transformation, we *apply* it to both sides of the equation i.e. we have to simplify:

$$3x-6 = 15$$

Now we have to choose another equivalence transformation, namely +6:

$$3x-6 = 15 \qquad | +6$$

And we simplify again:

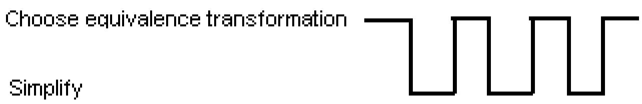
$$3x = 21$$

We are interested in the *practice* of teaching mathematics. In particular we want to know why students make what errors. A typical mistake starts with the following argument: “*There is a 3 in front of the variable x. To get rid of the 3 I need to subtract 3*”. This student is most likely to write ...

$$3x = 21 \qquad | -3$$
$$x = 18$$

..., believing that the equation is solved.

What goes wrong and how can technology help to make it better? An analysis of the steps taken above reveals two alternating tasks: (1) the choice of an equivalence transformation and (2) the simplification of algebraic expressions. Here, the choice of an equivalence transformation is a higher-level task insofar as it is the essence of the strategy for finding the solution of an equation. It is the new skill which the student has to learn when learning to solve equations. The simplification of expressions is a lower-level task, for which the teacher has to assume that the student is sufficiently well trained.

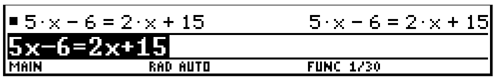


This picture demonstrates that a student, while trying to learn a new skill, repeatedly has to interrupt the learning process in order to perform a calculation. This is as if one would repeatedly be interrupted during a difficult chess game. In fact, it is even worse, because the interruption can influence the “game”: A mistake made during the interruption, i.e. during the lower-level task, severely disturbs the higher-level task and may prevent the student from learning. This is exactly what leaded to the wrong solution *x*=18 in the above example: After deciding to subtract 3, the student should fully concentrate on subtracting 3 from both sides of the equation while “forgetting” the reason for choosing this equivalence transformation. But, in reality, the student starts the next line with “*x*=” simply “*because the transformation –3 was chosen in order to generate x= on the left hand side*”. At the higher level the student has the (wrong) impression that –3 simplified the equation as desired.

This continuous change of levels inevitably occurs in almost all topics in school mathematics. It appears to be one of the central problems in mathematics education that students have to learn a new ability/skill while still practising an “old” one.

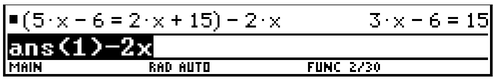
Using an algebraic calculator the learning process could be conducted as follows. First we enter the equation.

5x-6=2x+15 (ENTER)



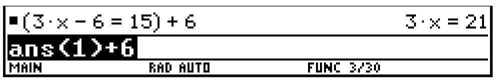
Then follows the input of the equivalence transformation. (The calculator automatically applies the subtraction operator to the last expression, i.e. the equation as its first argument.)

-2x   (ENTER)



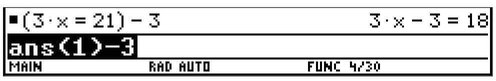
The simplification, i.e. the application of the equivalence transformation to both sides of the equation, was performed by the calculator. Then the student chooses the next equivalence transformation:

+6   (ENTER)



We mimic a student who makes the above discussed mistake:

-3   (ENTER)



It goes without saying that the calculator simplifies properly, hence the student receives an <u>immediate</u> feedback that the transformation –3 was not successful (i.e. did not simplify the equation to “*x*=”).

Here we meet again two issues that we discussed earlier: (1) The student *experiments* with possible equivalence transformations, hence there is an experimental learning phase, and (2) the *immediacy* of the result of applying the transformation tallies with what we asked for in the section on visualisation.

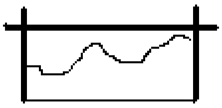
A student who follows the above calculator-supported exercise can fully concentrate on the (higher-level) skill of choosing an equivalence transformation. The lower-level skill of simplification is performed (at least for the moment) by the calculator.

In [Kutzler 1998a] I give detailed instructions how to use a TI-92 (or TI-89) to teach/learn solving of linear equations. The above algebraic approach is discussed in further detail as well as numeric and graphic approaches.

The Scaffolding Method

In the above subsection on concentration we compared teaching mathematics with building a house. In the language of this metaphor the above-mentioned problem of mathematics education translates into the problem of building a new storey on top of an incomplete storey. For example, as soon as we start building the storey of “choosing equivalence transformations”, the storey of “simplifying” is still incomplete for many of our students. In mathematics teaching at school we simply don’t have enough time for waiting until all students have completed all previous storeys. The curriculum forces the teacher to continue with the next topic, independent of the progress of individual students. So, it remains to ask how a student can build a storey on top of an incomplete one.

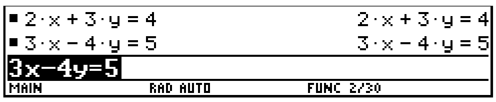
The above example demonstrates how I suggest answering this question: While the student learns the higher-level skill, the calculator solves all sub-problems that require the lower-level skill. Using the language of the metaphor, the calculator is scaffolding above the incomplete storey.



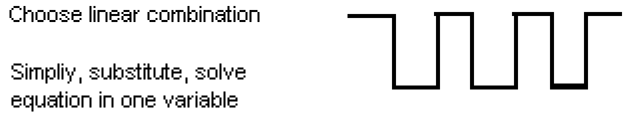
Using the example of solving a linear equation we demonstrated the use of an algebraic calculator as scaffolding above the simplification storey. In the sequel we apply the scaffolding method to another example, namely the solving of a system of linear equations with Gaussian elimination.

We use the system $2x+3y=4$, $3x-4y=5$. First, the student enters the equations:

2x+3y = 4   (ENTER)
3x-4y = 5   (ENTER)

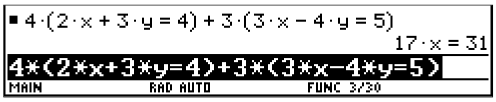


Gaussian elimination requires us to choose a linear combination of the two equations such that one variable is eliminated. This is what needs to be learned. Everything else (simplification, substitution, solving of an equation in one variable) are prerequisites of which the teacher expects that the students are well enough trained.



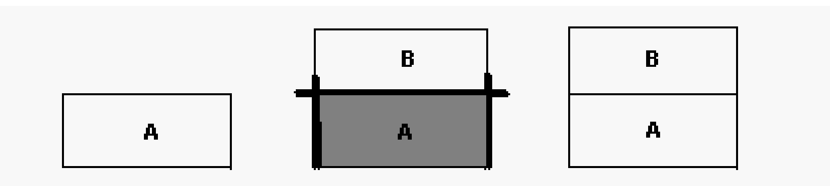
We try to eliminate *y* by adding four times the first equation and three times the second equation. (On both the TI-92 and the TI-89 there is a simple technique for auto-pasting a previously entered equation into the entry line. We don't give the details here.)

4*(.... first equation ...)+3*(... second equation ...)   (ENTER)



Voila. Variable *y* disappeared as requested. What is the practice of teaching this topic at school? Some students choose the right linear combination, but, due to a calculation error, the variable does not disappear. Other students choose a wrong linear combination, but, again due to a calculation error, the variable disappears (because it "must" disappear.) For both groups of students their weakness in algebraic simplification is a stumbling block for successfully learning the basic technique of Gaussian elimination. Exactly those students lag behind more and more the "higher" they get in the house of mathematics.

In the above exercise, the algebraic calculator is scaffolding that compensates any weakness with the lower-level skills, hence it helps avoiding mistakes. In case the final teaching goal is to have students be able to solve systems of equations (or perform any other skill B) manually (because, for example, the students are assessed centrally at the end of the school year), then it is recommended to perform the following three steps. The first step is teaching and practising skill A. The second step is teaching and practising skill B, while using the algebraic calculator to solve all those sub-problems that require skill A (i.e. the student can fully concentrate on learning skill B.) The third step is to combine skills A and B with no support from technology.



This is one of many conceivable methods of using algebraic calculators and computers as pedagogical tools.

The temporary use of technology can help to break down the learning process into smaller, easier "digestible" pieces. For less gifted students, who could not swallow the "big pieces" we offered them so far, this may be the only way of mastering these learning steps. They find it easier to keep track of the steps without getting lost (or screwed up) in details such as simplification. When comparing the teaching of mathematics with building a house, the use of technology compares with using scaffolding.

The ***scaffolding method*** is any pedagogically justified sequence of using and not using technology for trivialisation, experimentation, visualisation, or concentration either in the sense of automation or compensation.

The presented use of technology as a pedagogical tool is completely independent whether or not technology may be used during an exam. The scaffolding method aims at supporting the learning process, i.e. it can help reaching (traditional) teaching goals. Here, technology is only a training instrument. Like a home trainer can help us acquiring physical skills, an algebraic calculator can help us acquiring intellectual/mathematical skills. Consequently, technology should be introduced as a pedagogical tool independently of any changes to the curriculum or the assessment scheme. Algebraic calculators can help at all levels of mathematics in secondary education.

In [Kutzler 1998b] I describe how one can use a TI-92 (or a TI-89) to treat the topic "solving of systems of linear equations" at school. Besides giving further details on the above approach, the booklet also describes numeric and graphic methods as well as the substitution method.

Goals of mathematics education

I believe that the two central goals in mathematics education should be to train the students in the two disciplines *intellectual sports* and *problem solving*.

a) Intellectual sports

Given the existence of algebraic calculators, Bruno Buchberger raised the question *"Why Should Students Learn Integration Rules?"* ([Buchberger 1989]) and Wilfried Herget asked *"How Much Simplification Do We Need?" ("Wieviel Termumformung braucht der Mensch?"),* see the remark on page 8 in [Hischer 1992]. In all areas of life we need to ask how far we should go in automation.

At the beginning we compared mathematics with moving/transportation. Our transportation technology is far enough developed so that we would not need to walk at any time in our life. We could use moving tools from infancy on. But we don't do it. We know, that this would be devastating for our physical fitness and health: Our muscles would – because never used – degenerate, and this would certainly effect the whole body.

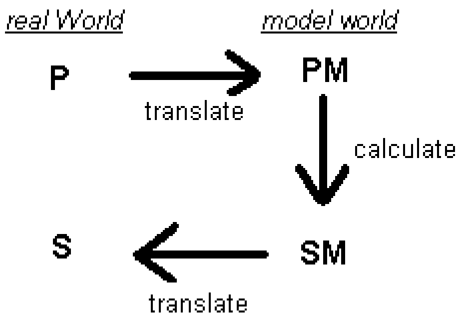
Due to a massive increase of automation over the past years, many of our intellectual skills are in jeopardy. In the past we needed to memorise phone numbers – today we just use the phone's memory keys. In the past we had to memorise how to program the video recorder – today we just swipe a bar code reader over the TV programme. It goes without saying that all this makes life so much more comfortable. BUT it leads us to loosing what I call "intellectual fitness". Many teachers complain about students' lack of concentration and their weak memories. These are two typical symptoms for a diminishing intellectual health.

In medicine there exist definitions of what a "healthy" person must be able to do physically. After a heart attack, for example, a patient must be able to walk a certain distance and to take the stairs a certain number of levels before he is considered cured. We need something similar for our intellectual capabilities, i.e. a definition of what an "intellectually healthy" person must be able to do in terms of, for example, memorisation, or mental calculations.

For what concerns our physical capabilities, we introduced the school subject "sports" in order to fight a further deterioration. We need to take similar steps regarding our intellectual capabilities, e.g. we need the introduction of a school subject "intellectual sports". I believe that this should be ***one*** of the goals of mathematics education.

b) Problem solving

In mathematics I consider problem solving to be the ability to use mathematical tools for solving real world technical problems. Characteristic for problem solving are the three steps shown in the graphics below. The first step is choosing the model and translating the real world problem into the language of the model, which requires us to grasp and understand the problem. (An optimisation problem, for example, would translate into a function to be optimised and equations which describe any constraints between the variables involved.) The second step is applying the available algorithms to solve the model problem PM, yielding a model solution SM. The third step, finally, is to translate the model solution into a real world solution S. (Often, people refer to this re-translation as interpretation.)



However, now we still need to test, if S actually is a solution of P. In case it is not, then the whole process (i.e. all three steps) need to be repeated, because the mistake or error could be anywhere: The chosen model may be inappropriate, the translation may be faulty, or the calculation may be wrong.

Today, problem solving is treated at school only half-heartedly. The main emphasis is put on the second step, calculation, and its execution with paper and pencil. Therefore, most problem solving exercises turn into exercises for practising calculation skills. Since translation hardly is taught explicitly, it is understandable that a majority of students don't develop this ability, hence they are afraid of this type of exercises. As a consequence, most students believe that such exercises are only for the most ingenious among them.

By employing technology as widely as possible, we can dedicate enough time to teach the choosing of models and the translating. Once these skills are taught explicitly, more students will appreciate and master them.

c) The future of teaching mathematics

Curricula should aim at educating students in the disciplines *intellectual sports* and *problem solving*.

The goal of intellectual sports are intellectual fitness and health as has been described in detail earlier. It goes without saying that technology should only be a training tool ("mathematical home trainer") and must not be used when testing these abilities.

The goal of problem solving is the ability of using given tools for solving given problems. It goes without saying that the use of technology should be highly welcome here.

Once again I draw a parallel using the initial comparison of mathematics with moving/transportation. Problem solving compares with an actual desire of moving from A to B (or transporting something from A to B). Only the reaching of B counts. It is less important (maybe even unimportant) how we got there. Intellectual sports compares with a jogger who runs along a track in order to gain physical fitness. Only the jogging counts. It is unimportant where the track is or where it leads to. Transferred to mathematics this means the following: In problem solving only the result counts. It is irrelevant how the calculations were performed. In intellectual sports only the performing of the calculation counts, while the result is unimportant.

d) Assessment

The above yields a very simple rule for assessment: When assessing intellectual fitness, no tool is allowed, not even a simple four-function calculator. When assessing problem solving, all tools are allowed (better: solicited), in particular graphic or algebraic calculators. In case this splitting is not manageable within an exam, one should assess the two disciplines at different times. It is obvious to draw a parallel with ice-skating: Intellectual sports compare with the compulsory exercise, in which the athlete demonstrates the mastery of the basic techniques. Problem solving compares with the voluntary exercise (freestyle), in which the athlete demonstrates the ability to combine the basic techniques into a nice choreography. The total score depends on the scores of both the compulsory and the voluntary part.

*

In the end technology should play a secondary role in both disciplines. In intellectual sports the goal is a performance with a minimum of tools. In problem solving the goal is to learn all those skills and abilities that are needed for problem solving and that are not supported by any technology. A good mathematics education will use a calculator or computer like language education uses dictionaries.

Concluding remarks

William Shakespeare once said: *"Nothing is either good or bad – only thinking makes it so."* Looking at technology for (mathematics) teaching one may change this into: *"Calculators and computers are neither good nor bad teaching tools – only using makes them so."* When driving a car, the most important is the driver – the car is secondary. Similarly, when teaching with technology the most important is the teacher – technology comes second. This is another plea for strengthening teacher preservice and inservice training.

In Austria in 1991 all general high schools (Gymnasien) and technical high schools (Höhere Technische Lehranstalten) were equipped with the DERIVE computer algebra system. In the sequel a research project was conducted which became known as the "Austrian DERIVE Project". The project involved 800 students who were taught regular mathematics with DERIVE. The results were published in the German language book [Heugl/Klinger/ Lechner 1996] and can also be found in the English language publication [Aspetsberger/Fuchs 1996]. In the academic year 1997/98 the same team carried out the "Austrian TI-92 Project" with 2,000 students using the TI-92 in regular mathematics classes. The results can be found in the internet at the address http://www.acdca.ac.at. Currently another TI-92 project is on involving 3,000 students.

The Austrian and other investigations showed the following: If technology is used properly, it leads to

- more efficient teaching and learning,
- more independent productive student activity,
- more student creativity,
- an increased importance of the teacher.

The teacher has the duty to accompany and direct the students at their partly individual voyage of discovery through the world of mathematics. Consequently the key to the success in teaching mathematics is good teacher training. Not technology changes teaching, but technology is a catalyst for teachers to change their teaching methods and focus on topics and skills, aiming at a better teaching of mathematics.

In case you have questions or suggestions, please write to me at b.kutzler@eunet.at. A regularly updated collection of information about technology in mathematics education can be found at www.kutzler.com.

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