Fourth International Derive TI-89/92 Conference
Liverpool John Moores University, July 12-15, 2000

Calculators and Spreadsheets – All Together Now?
A Workshop Using the TI-92

David Bowers
Suffolk College, U.K.
Email: david.bowers@suffolk.ac.uk

Foreword
Spreadsheets have long been a useful tool for analysing problems numerically. More recently, computer algebra systems have hit the scene. We demonstrate how the Texas Instruments TI-92 goes some way to incorporate computer algebra into a simple spreadsheet environment. Examples of some unexpected applications will be investigated.

Firstly the basic functions and operations of the Data/Matrix Editor are reviewed. Then various activities covering a range of mathematical areas are proposed. The first three of these deal with numerical applications of the type that may be familiar to users of spreadsheets in the mathematics classroom, and serve to illustrate the spreadsheet-style use of the Data/Matrix Editor. The subsequent activities introduce some features of computer algebra systems that hitherto have not been available within a spreadsheet environment.

Participants are encouraged to review the activities and their usefulness or otherwise for demonstrating and investigating the mathematical concepts, to consider the advantages and technical limitations of the Data/Matrix Editor for this purpose, and to speculate on the future development of algebraic spreadsheets.

===============

Fourth International Derive TI-89/92 Conference

Contents

Introducing the Texas TI-92 Data/Matrix Editor

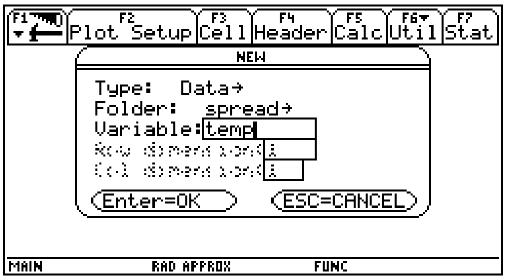
Before we start, let us create a new Folder called "spread" for our work:

VAR-LINK
F1: Manage
5: Create Folder and enter the name spread

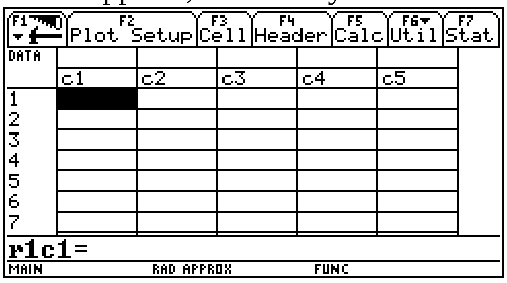
The Data/Matrix editor is accessed by pressing the big blue APPS key:

APPS
6: Data/Matrix Editor
3: New…

and in the resulting dialog box call the new variable *temp* as shown:



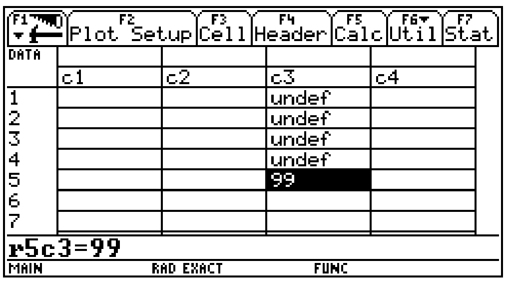
An empty data table appears, which may look something like this:



The width of the columns can be changed:

F1:
9: Format…
Cell Width: 8 *(this fits four columns in screen. Experiment with others)*
Auto-calculate: ON *(we will keep this setting throughout)*

Data is entered in columns. Each column is a "list" (regular users of Texas calculators will understand the significance of lists). Use the cursor pad to move around the cells of the table. Note the name of the cell in the entry line, in the form **r1c1=**
Type the cell content in the entry line, then press ENTER or the down cursor. See what happens if you try to enter something directly into cell r5c3, for example:



This illustrates that the Data/Matrix Editor thinks in complete columns (which can be up to 999 elements long). It is ***not possible*** to specify the content of an individual cell in terms of another individual cell, as you can with standard spreadsheets. Try to get the value 49.5 in r6c4 by defining it in the entry line as **r6c4 = r5c3 / 2**. No luck!

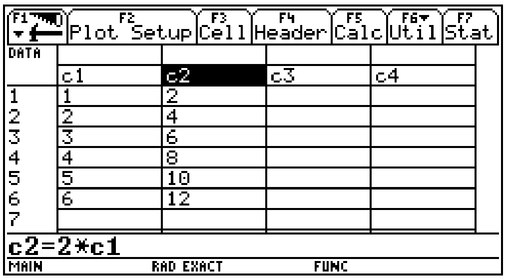
What you ***can*** do, and this is the key feature for using the Data/Matrix Editor as a simple kind of spreadsheet, is to ***define one column as a function of others***. Let us try this. First clear the current screen by

F1:
8: Clear Editor

Enter the values 1, 2, 3, 4, 5, 6 in c1
Now highlight the column header c2 for the second column. You can now define column c2 as a function of column c1. Try, for example

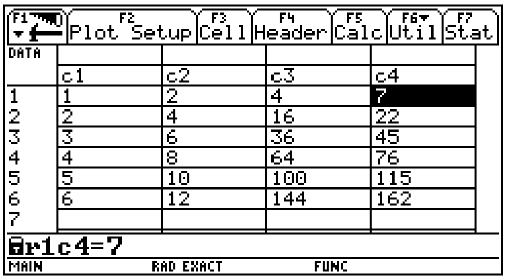
c2 = 2*c1

You should see the following screen, as expected. Note that the length of column c2 is dictated by the length of column c1:



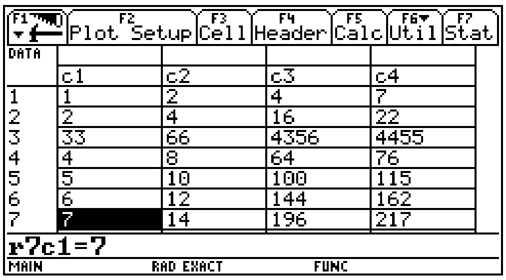
Now define columns c3 and c4, for example by

c3 = c2^2 and **c4 = c1 + c2 + c3**



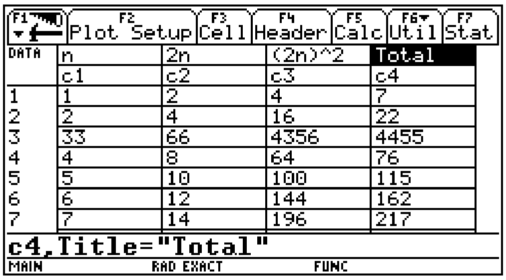
Note that when you highlight a cell in a column which has been defined as a function of other columns, a padlock appears against its name in the entry line. This shows that the content of that cell is "locked" by the column definition and cannot be altered.

Of course, you can alter the contents of column c1. Note that, if you alter a value, the corresponding values in the other columns change. Also, if you add an element, the other columns increase in



length accordingly:

The blank cells above the column headers can be used for column titles. Move the cursor to the cell, and type in whatever seems appropriate:



DATA	n	2n	(2n)^2	Total
	c1	c2	c3	c4
1	1	2	4	7
2	2	4	16	22
3	33	66	4356	4455
4	4	8	64	76
5	5	10	100	115
6	6	12	144	162
7	7	14	196	217

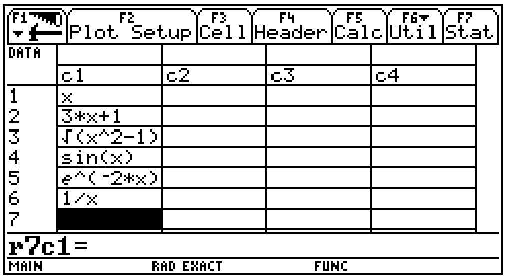
c4,Title="Total"

You can clear the contents of the Data/Matrix Editor now by pressing

F1:

8: Clear Editor

The facility of the Data/Matrix Editor that we want to focus on here is the ability to define data and column definitions symbolically. Enter yourself a variety of functions in column c1, for example:

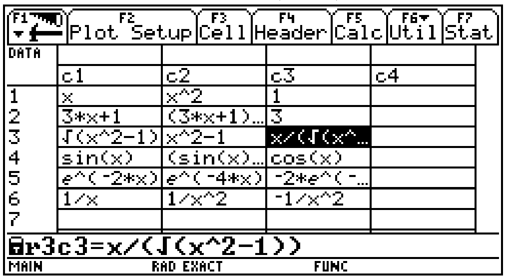


DATA	c1	c2	c3	c4
1	x			
2	3*x+1			
3	√(x^2-1)			
4	sin(x)			
5	e^(-2*x)			
6	1/x			
7				

r7c1=

Now define subsequent columns as functions of c1:

c2 = c1^2 and **c3 = d(c1,x)** the derivative of c1



DATA	c1	c2	c3	c4
1	x	x^2	1	
2	3*x+1	(3*x+1)…	3	
3	√(x^2-1)	x^2-1	x/(√(x^…	
4	sin(x)	(sin(x)…	cos(x)	
5	e^(-2*x)	e^(-4*x)	-2*e^(-…	
6	1/x	1/x^2	-1/x^2	
7				

r3c3=x/(√(x^2-1))

We note two things:

1) This "algebraic spreadsheet" does not use "pretty print".
2) Often the columns are too narrow to contain the full expression. This is shown by the … at the end. Using the cursor to highlight a cell shows its content in the entry line (although, as we will see, this does not always help). Of course, we could increase the width of the columns, but the maximum width available is only 12 characters.

The reader may now wish to experiment further with the Data/Matrix Editor, or move on to the structured activities overleaf.

Fourth International Derive TI-89/92 Conference

Activity 1: A simple business spreadsheet

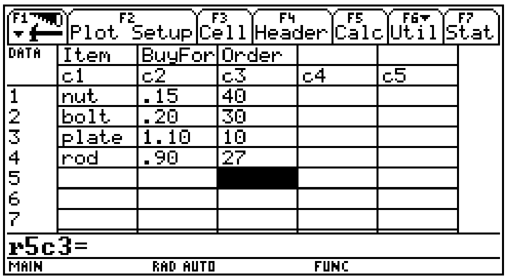
Start a new datasheet called "business" in your "spread" folder:

APPS
6: Data/Matrix Editor — *we shall assume*
3: New… — *you can do this*
Type: Data — *in future!*
Folder: spread
Variable: business

Change the cell width to 6 (to get five columns on screen) and check auto-calculate on

F1:
9: Format… — *and this!*

We shall set ourselves up as ironmongers, and order in some stock. Enter the business data as shown, including the column titles:



DATA	Item	BuyFor	Order		
	c1	c2	c3	c4	c5
1	nut	.15	40		
2	bolt	.20	30		
3	plate	1.10	10		
4	rod	.90	27		
5					
6					
7					

r5c3=

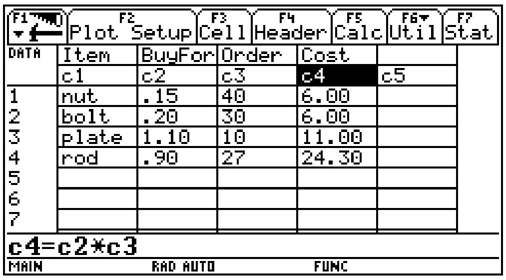
MAIN RAD AUTO FUNC

(Hint: Use **Display Digits = Fix2** *from the* **MODE** *options)*

What is the total cost of our order? We define column c4 as the costs of each item:

c4 = c2 * c3

and obtain the order costs of each item:

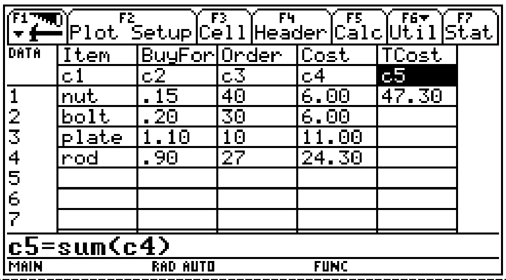


DATA	Item	BuyFor	Order	Cost	
	c1	c2	c3	c4	c5
1	nut	.15	40	6.00	
2	bolt	.20	30	6.00	
3	plate	1.10	10	11.00	
4	rod	.90	27	24.30	
5					
6					
7					

c4=c2*c3

MAIN RAD AUTO FUNC

The obvious thing to do now is to find the total order cost as a column total. In a standard spreadsheet, we would do this by highlighting cell r5c4 and typing something like **r5c4=sum(c4)**. Try this now - no luck! You cannot edit the contents of any cell in c4 because the column is "locked" by its definition. What you have to do here is to define c5 as the sum of c4: **c5=sum(c4)**



DATA	Item	BuyFor	Order	Cost	TCost
	c1	c2	c3	c4	c5
1	nut	.15	40	6.00	47.30
2	bolt	.20	30	6.00	
3	plate	1.10	10	11.00	
4	rod	.90	27	24.30	
5					
6					
7					

c5=sum(c4)

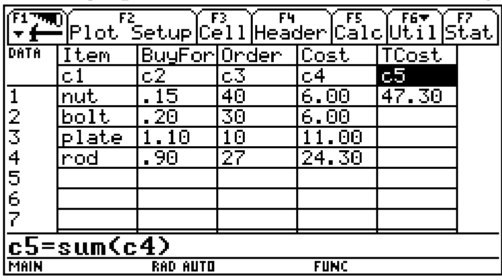
MAIN RAD AUTO FUNC

Fourth International Derive TI-89/92 Conference

Now check yourself that changing any item cost in c2 and/or any order quantity in c3 will result in the corresponding values in c4 and the total cost in c5 being updated automatically. (Simply use the cursor pad to highlight which cell you want to change, and type the new value.)

We can now extend the spreadsheet to include our selling prices, and get the spreadsheet to automatically calculate our profit.

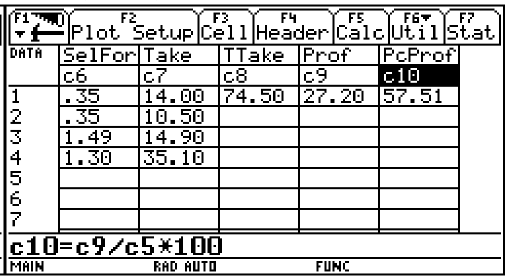
Enter some appropriate selling prices into column c6. Under the assumption that we will sell all the items we stock, define c7 to be the take and define **c7=c3*c6**.
The total take can be shown in c8 by **c8=sum(c7)**
Now what about our profit on this business?
The actual profit can be shown in c9 by **c9=c8-c5**
The percentage profit can be shown in c10 by **c10=(c9/c5)*100**



F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	Item	BuyFor	Order	Cost	TCost
	c1	c2	c3	c4	c5
1	nut	.15	40	6.00	47.30
2	bolt	.20	30	6.00	
3	plate	1.10	10	11.00	
4	rod	.90	27	24.30	
5					
6					
7					

c5=sum(c4)
MAIN RAD AUTO FUNC



F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

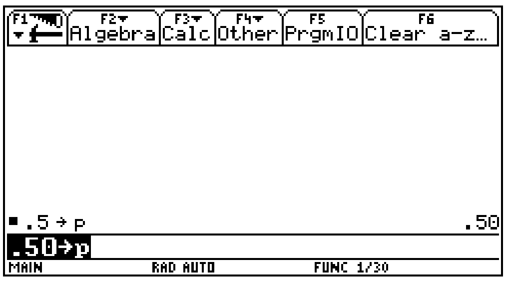
DATA	SelFor	Take	TTake	Prof	PcProf
	c6	c7	c8	c9	c10
1	.35	14.00	74.50	27.20	57.51
2	.35	10.50			
3	1.49	14.90			
4	1.30	35.10			
5					
6					
7					

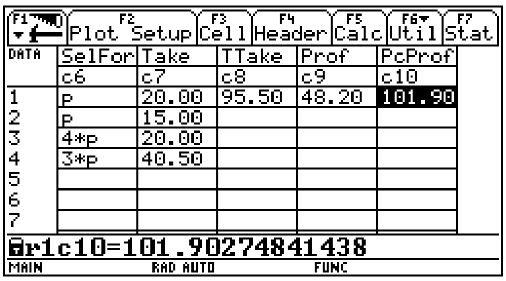
c10=c9/c5*100
MAIN RAD AUTO FUNC

We note that we soon hit upon the limitations of the TI-92's screen size, and need to scroll backwards and forwards to keep track of what we are doing. However, we have created a simple spreadsheet with similar functionality to a "real" one, in that a full "what if?" analysis can be carried out by varying the costs, order numbers and selling prices (c2, c3 and c6) and observing the effect on the profit.

Now comes something useful……

We recall that **a function or variable defined in the Home screen of the TI-92 retains its value in the Data/Matrix Editor**. So suppose in this example, we wish to keep things simple by making the selling prices of nuts and bolts the same, the selling price of a plate 4 times that of a nut and the selling price of a rod 3 times that of a nut. This would mean entering *p, p, 4p* and *3p* into column c6. Note that subsequent columns which depend on c6 are recalculated in terms of *p*. Unfortunately, since their contents are locked, we cannot work with them directly. However, let us vary the value of *p*.
Jump to the Home screen **[♦] HOME** and assign *p* the value 0.50 by **0.50 STO> p** .
Now revert to the current spreadsheet by **APPS 6:Data/Matrix Editor 1:Current** and observe the updated values:





F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	SelFor	Take	TTake	Prof	PcProf
	c6	c7	c8	c9	c10
1	p	20.00	95.50	48.20	101.90
2	p	15.00			
3	4*p	20.00			
4	3*p	40.50			
5					
6					
7					

r1c10=101.90274841438
MAIN RAD AUTO FUNC

Can you guess a value of *p* which will yield a percentage profit of (say) 80%?

Activity 2: Solving an equation

This is a classic application of spreadsheets for solving mathematical problems. A solution of *f(x) = 0* is found by tabulating values of *f(x)* over an initial discrete domain, and observing an interval where the function changes sign. This interval is then tabulated in smaller steps, and the process repeated until the desired accuracy reached.

This procedure is particularly easy to apply on the TI-92.

For example, let us find the root of the equation $x^3 + x - 100 = 0$.

Start a new datasheet called "eqnsolve" in your "spread" folder, and choose cell width 12.

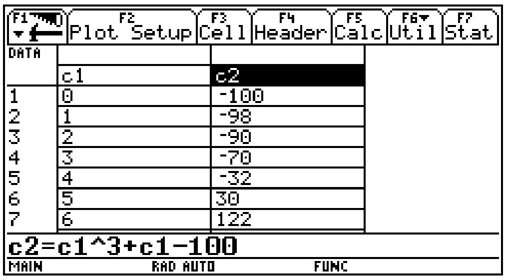
In this and subsequent Activities, we shall make use of the SEQ( ) function. Its syntax is: ***SEQ(expression , variable , low , high [, step])***
and it will automatically fill a column with a sequence. Default step is 1.

To set up an initial list of values for *x*, use the SEQ( ) function as follows:

c1 = SEQ(i , i , 0 , 6)

The corresponding values of the function can be tabulated in column c2 by:

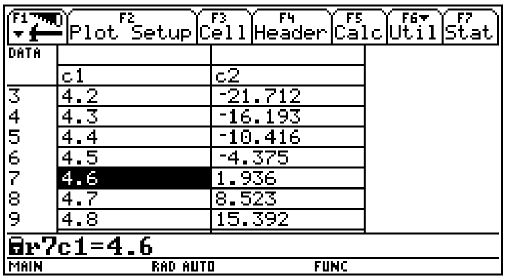
c2 = c1^3 + c1 - 100



Thus we see that there is a change of sign between *x = 4* and *x = 5*. So we re-tabulate within this interval, to 1 decimal place now. Do this simply by editing the column heading of c1 to read:

c1 = SEQ(i , i , 4 , 5 , 0.1)

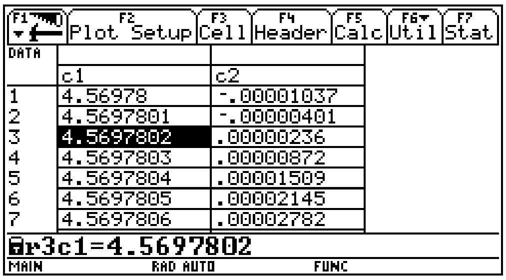
The function values in c2 are updated automatically. (HINT: If you still have the display set to FIX2 from the previous activity, change it now to FLOAT8). By scrolling down, we see now that the root must lie between *x = 4.5* and *x = 4.6*



Repeated editing of the definition on the values in column c1 allows us to "zoom in" on the position of the root of the equation.

Fourth International Derive TI-89/92 Conference

After a few more iterations, we may end up with a table like the one below. (HINT: to edit the column header, you do not need to scroll back up to the top of the column. Simply press F4.)



	c1	c2
1	4.56978	-.00001037
2	4.5697801	-.00000401
3	4.5697802	.00000236
4	4.5697803	.00000872
5	4.5697804	.00001509
6	4.5697805	.00002145
7	4.5697806	.00002782

Thus the solution would appear to be $x = 4.569780$ correct to 6 d.p.

Of course, the integrated nature of the TI-92 as a tool for doing mathematics means that this solution can be verified by using the SOLVE( ) function in the Home screen, or by plotting and zooming in on the graph of $y = x^3 + x - 100$ where it cuts the x-axis. It is this interplay between numerical, analytical and graphical approaches to problem-solving which can be supported by the new educational technology now available.

TASK: Use the above spreadsheet technique to locate the root of the equation

$$x^3 + 3x^2 = 75$$ accurate to 5 d.p.

and verify using alternative approaches.

Activity 3: Locating a turning point

A spreadsheet approach similar to that of locating a root in the previous activity can be used to locate a turning point of a function, by progressively "zooming in" on the local maximum (or minimum) value and re-tabulating to greater accuracy.

For example, locate the maximum value of the function $f(x) = x\,e^{-3x}$

Start a new datasheet called "tpoint" in your "spread" folder.

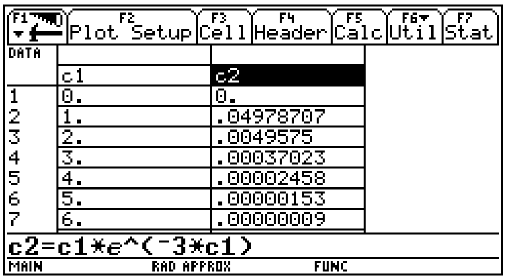
Set up an initial domain of x-values in column c1:

c1 = SEQ(i , i , 0 , 6)

and define column c2 as the given function:

c2 = c1 * e^(-3*c1)

Note that you should use MODE to set results to approximate mode, in order to work numerically:



F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA

	c1	c2
1	0.	0.
2	1.	.04978707
3	2.	.0049575
4	3.	.00037023
5	4.	.00002458
6	5.	.00000153
7	6.	.00000009

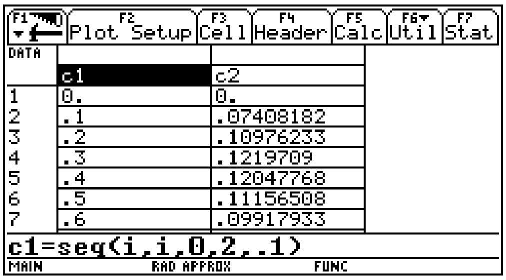
c2=c1*e^(⁻3*c1)

MAIN RAD APPROX FUNC

Here, we can observe that the function achieves its maximum somewhere within the "double-width interval" between $x = 0$ and $x = 2$. Zooming in on this by editing the column header of c1 to

c1 = SEQ(i , i , 0 , 2 , 0.1)

yields the new table



F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

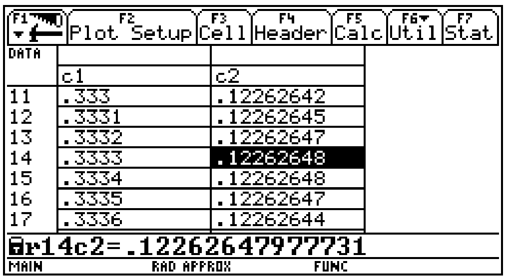
DATA

	c1	c2
1	0.	0.
2	.1	.07408182
3	.2	.10976233
4	.3	.1219709
5	.4	.12047768
6	.5	.11156508
7	.6	.09917933

c1=seq(i,i,0,2,.1)

MAIN RAD APPROX FUNC

and repeating this process leads eventually to a table something like:



F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA

	c1	c2
11	.333	.12262642
12	.3331	.12262645
13	.3332	.12262647
14	.3333	.12262648
15	.3334	.12262648
16	.3335	.12262647
17	.3336	.12262644

r14c2=.12262647977731

MAIN RAD APPROX FUNC

where we would deduce the approximate co-ordinates of the T.P as (0.333 , 0.1226265)

Fourth International Derive TI-89/92 Conference

There is an alternative approach to locating turning points using a spreadsheet, based on the fact that the gradient changes sign on either side of the turning point.

Suppose *f(x)* is given for a discrete set of equispaced ordered points $\{x_1 , x_2 , x_3 , \ldots\}$.
Then if $f(x_n) - f(x_{n-1})$ and $f(x_{n+1}) - f(x_n)$ are of different signs, then *f* has a turning point for some *x* between x_{n-1} and x_{n+1} . (Assumptions of continuity are made.)

This can be implemented neatly using the SHIFT( ) function, which shifts the elements of a column a specified number of rows.

Start a new datasheet called "tpoint2" in your "spread" folder:

c1 = SEQ(i , i , 0 , 6)
c2 = c1*e^(-3*c1)
c3 = SHIFT(c2 , 1)
c4 = c2 - c3

DATA	x	f(x_n)	f(x_n+1)	slope
	c1	c2	c3	c4
1	0.	0.	.0497871	-.049787
2	1.	.0497871	.0049575	.0448296
3	2.	.0049575	.0003702	.0045873
4	3.	.0003702	.0000246	.0003457
5	4.	.0000246	.0000015	.000023
6	5.	.0000015	9.138E-8	.0000014
7	6.	9.138E-8	undef	undef

c4=c2-c3

By considering the change of sign in the slope, we identify that a turning point must be located between *x* = *0* and *x* = *2*. We edit column c1 accordingly:

DATA	x	f(x_n)	f(x_n+1)	slope
	c1	c2	c3	c4
1	0.	0.	.0740818	-.074082
2	.1	.0740818	.1097623	-.035681
3	.2	.1097623	.1219709	-.012209
4	.3	.1219709	.1204777	.0014932
5	.4	.1204777	.1115651	.0089126
6	.5	.1115651	.0991793	.0123857
7	.6	.0991793	.0857195	.0134598

c1=seq(i,i,0,2,.1)

The turning point is now identified as between *x* = *0.2* and *x* = *0.4*.
Continuing likewise, we end up eventually with a table something like:

DATA	x	f(x_n)	f(x_n+1)	slope
	c1	c2	c3	c4
9	.33328	.1226265	.1226265	-5.3E-10
10	.33329	.1226265	.1226265	-4.2E-10
11	.3333	.1226265	.1226265	-3.1E-10
12	.33331	.1226265	.1226265	-2.E-10
13	.33332	.1226265	.1226265	-9.2E-11
14	.33333	.1226265	.1226265	1.84E-11
15	.33334	.1226265	.1226265	1.29E-10

r14c1=.33333

from which we deduce, as before, that the turning point is at approximately
$x = 0.3333$, $y = 0.1226265$

TASK: Use a spreadsheet technique to find the turning points of the graph
$y = x^3 - 9x + 1$
Verify your results using analytical and graphical methods.

Activity 4: Investigating surds

The rules for manipulating surd quantities are often taken on trust. Computer algebra systems allow arithmetic to be done exactly, and answers can be given in simplified surd form. Combining this with the tabular approach of a spreadsheet, the student can generate a variety of examples quickly, and investigate the application of the underlying rules.

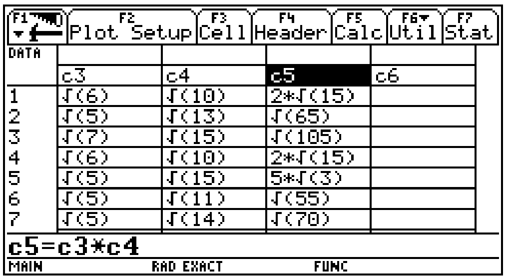
We focus here on the result **√m * √n = √mn** where √mn may simplify further.

Start a new datasheet called "surds" in your "spread" folder.

We will make use here of the function RAND(*n*), which generates a random integer between 1 and n for a positive integer *n*. Therefore a random integer between 5 and 8 inclusive will be given by RAND(4) + 4, and a random integer between 10 and 15 inclusive will be given by RAND(6) + 9. These two sets of numbers will generate appropriate data for our investigation.

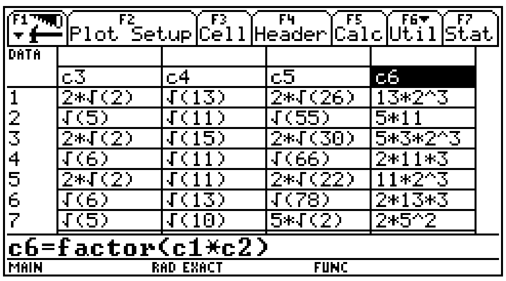
Firstly, set the MODE to exact. Then:

c1 = SEQ(RAND(4) + 4, i, 1, 20)
c2 = SEQ(RAND(6) + 9, i, 1, 20)
c3 = √(c1)
c4 = √(c2)
c5 = c3 * c4



	c3	c4	c5	c6
1	√(6)	√(10)	2*√(15)	
2	√(5)	√(13)	√(65)	
3	√(7)	√(15)	√(105)	
4	√(6)	√(10)	2*√(15)	
5	√(5)	√(15)	5*√(3)	
6	√(5)	√(11)	√(55)	
7	√(5)	√(14)	√(70)	

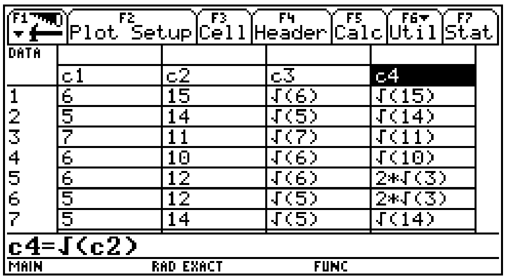
Scrolling down the lists, the student should justify the values in column c5 on the basis of the values in columns c3 and c4. Sometimes it is "obvious" - as in rows 2 and 3 in the screen-dump above - but sometimes it needs further explanation - as in rows 4 and 5. *Note that a fresh screen of data is automatically generated if you highlight and enter any column header (which re-invokes the RAND( ) function).*
The connection with the factorisation of the product *mn* can be made clear by:



	c3	c4	c5	c6
1	2*√(2)	√(13)	2*√(26)	13*2^3
2	√(5)	√(11)	√(55)	5*11
3	2*√(2)	√(15)	2*√(30)	5*3*2^3
4	√(6)	√(11)	√(66)	2*11*3
5	2*√(2)	√(11)	2*√(22)	11*2^3
6	√(6)	√(13)	√(78)	2*13*3
7	√(5)	√(10)	5*√(2)	2*5^2

c6 = FACTOR(c1*c2)

The student should be encouraged to produce a table of results similar to the one above, and to explain each of the values in column c5. As a self-test, the student could scroll back across to "hide" columns c5 and c6, and generate a new table like the one below, then work out what s/he thinks the values in c5 will be given as:



	c1	c2	c3	c4
1	6	15	√(6)	√(15)
2	5	14	√(5)	√(14)
3	7	11	√(7)	√(11)
4	6	10	√(6)	√(10)
5	6	12	√(6)	2*√(3)
6	5	12	√(5)	2*√(3)
7	5	14	√(5)	√(14)

c4=√(c2)

TASK: Logarithms also have underlying rules which students often find difficult to internalise. For example:

log(ab) = log(a) + log(b)

Produce a spreadsheet to demonstrate these rules, which will allow students to investigate their properties.

Activity 5: Summing series

Spreadsheets are ideal tools for generating sequences and series, and to investigate sums and limits. We shall consider how the TI-92 can deal with this for numerical and algebraic series.

Start a new datasheet called "sumn" in your "spread" folder.

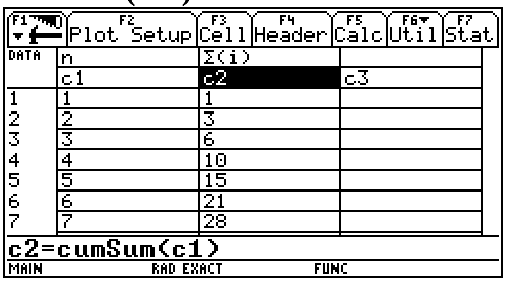
We firstly attempt to "discover" the result $\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$

Generate the terms of the series in column c1 by

c1 = SEQ(i , i , 1 , 20)

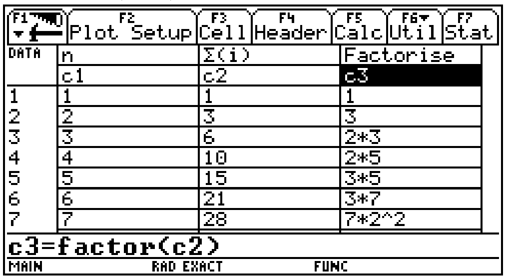
and then calculate the cumulative sums of this series in column c2 by

c2 = CUMSUM(c1)



Now factorise the values in column c2:

c3 = FACTOR(c2)



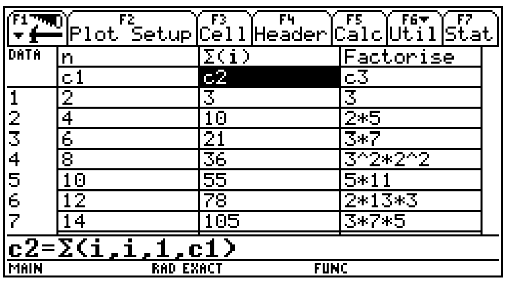
The results seem to be going up in pairs:
1*1 , 1*3 ;   2*3 , 2*5 ;   3*5 , 3*7 ;   4*7 , 4*9 ; ….

Perhaps it would be worth considering the even-numbered sums and the odd-numbered sums separately. This necessitates re-structuring the spreadsheet:

c1 = SEQ(2*i , i , 1 , 10)

c2 = Σ(i , i , 1 , c1) *Note the difference*

c3 = FACTOR(c2) *between Σ( ) and SUM( )*



Scrolling down the list, it appears that *n/2* is always a factor of the sum when *n* is even. The remaining factor(s) are 3, 5, 7, 9, 11, 13, 15 … which is a pretty clear indication of *(n+1)*. So the result *(n/2)(n+1)* appears to hold when *n* is even.

Now modify the spreadsheet (edit c1) to give the outcome when n is odd:

c1 = SEQ(2*i-1 , i , 1 , 10)

F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	n	Σ(i)	Factorise
	c1	c2	c3
1	1	1	1
2	3	6	2*3
3	5	15	3*5
4	7	28	7*2^2
5	9	45	5*3^2
6	11	66	2*11*3
7	13	91	7*13

c1=seq(2*i-1,i,1,10)

MAIN RAD EXACT FUNC

Here, it appears that *n* is always a factor of the sum when *n* is odd. The remaining factor(s) are 1, 2, 3, 4, 5, 6, 7 … which is a pretty clear indication of *(n+1)/2*.
So the result *(n)(n+1)/2* appears to hold when *n* is odd.

We are thus in a position to hypothesise that the formula *(n/2)(n+1)* will hold for any *n*. Of course, it is difficult for us as professional mathematicians to "unlearn" this standard result and discover it anew in this manner. However, students might appreciate the spreadsheet approach as a way of investigating the series.

TASK 1: "Discover" the results for $\sum_{i=1}^{n} i^2$ and $\sum_{i=1}^{n} i^3$

TASK 2: Hypothesise a formula for the sum of n terms of the series

$1.2^2.3 + 2.3^2.4 + 3.4^2.5 + \ldots + r.(r+1)^2.(r+2) + \ldots$

We will now consider the well-known arithmetic and geometric series.
Start a new datasheet called "apgp" in your "spread" folder.

Generate the sum of terms of an A.P. with first term *a* and common difference *d* **c1 = SEQ(i , i , 1 , 10)**

c2 = a + (c1 - 1)*d

c3 = CUMSUM(c2)

F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	n	T(n)	S(n)
	c1	c2	c3
1	1	a	a
2	2	a+d	2*a+d
3	3	a+2*d	3*a+3*d
4	4	a+3*d	4*a+6*d
5	5	a+4*d	5*a+10*d
6	6	a+5*d	6*a+15*d
7	7	a+6*d	7*a+21*d

c3=cumSum(c2)

MAIN RAD EXACT FUNC

If the pattern in the coefficients of *d* are not recognised, we could proceed as before by factorising columns c3:

c4 = FACTOR(c3)

F1	Plot Setup	Cell Header	Calc Util Stat
DATA	T(n)	S(n)	Factorise
	c2	c3	c4
1	a	a	a
2	a+d	2*a+d	2*a+d
3	a+2*d	3*a+3*d	3*(a+d)
4	a+3*d	4*a+6*d	2*(2*a+3*d)
5	a+4*d	5*a+10*d	5*(a+2*d)
6	a+5*d	6*a+15*d	3*(2*a+5*d)
7	a+6*d	7*a+21*d	7*(a+3*d)

c4=factor(c3)

MAIN RAD EXACT FUNC

As earlier, there seems to be a different underlying pattern for the even terms and the odd terms. So we could re-structure the spreadsheet as follows

c1 = SEQ(2*i , i , 1 , 10)

c2 = Σ(a + (i-1)*d , i , 1 , c1)

c3 = FACTOR(c2)

F1	Plot Setup	Cell Header	Calc Util Stat
DATA	n	S(n)	Factorise
	c1	c2	c3
1	2	2*a+d	2*a+d
2	4	4*a+6*d	2*(2*a+3*d)
3	6	6*a+15*d	3*(2*a+5*d)
4	8	8*a+28*d	4*(2*a+7*d)
5	10	10*a+45*d	5*(2*a+9*d)
6	12	12*a+66*d	6*(2*a+11*…
7	14	14*a+91*d	7*(2*a+13*…

c2=Σ(a+(i-1)*d,i,1,c1)

MAIN RAD EXACT FUNC

The pattern here seems clear: *n/2 * (2*a + (n-1)*d)*

Edit this now to consider the odd terms:

c1 = SEQ(2*i-1 , i , 1 , 10)

F1	Plot Setup	Cell Header	Calc Util Stat
DATA	n	S(n)	Factorise
	c1	c2	c3
1	1	a	a
2	3	3*a+3*d	3*(a+d)
3	5	5*a+10*d	5*(a+2*d)
4	7	7*a+21*d	7*(a+3*d)
5	9	9*a+36*d	9*(a+4*d)
6	11	11*a+55*d	11*(a+5*d)
7	13	13*a+78*d	13*(a+6*d)

c1=seq(2*i-1,i,1,10)

MAIN RAD EXACT FUNC

There is also a clear pattern here: *n * (a + ((n-1)/2)*d)*

Once it is appreciated that these two formulae are identical *(how could this be done by a weak student using the facility of computer algebra?)* then the well-known formula for the sum of an A.P. seems to have been "discovered".

TASK 3: "Discover" the formula for the sum of n terms of a geometric series.

Activity 6: Differentiation by first principles

Spreadsheets allow the generation of numerical sequences which tend to a limit. This can be incorporated with the algebraic facilities of the TI-92 to allow a visualisation of the limiting behaviour of the gradient function.

Start a new datasheet called "diff1" in your "spread" folder.

We wish to consider the limit of $\dfrac{f(x+h) - f(x)}{h}$ as h tends to zero.

In column c1 generate a sequence of values for h:

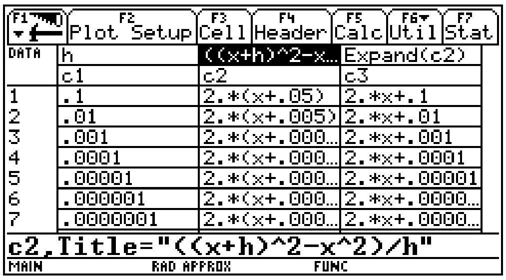
c1 = SEQ(0.1^i , i , 1 , 10)

Start off by considering the function $f(x) = x^2$

c2 = ((x + c1)^2 - x^2) / c1

and to make the final result clear

c3 = EXPAND(c2)

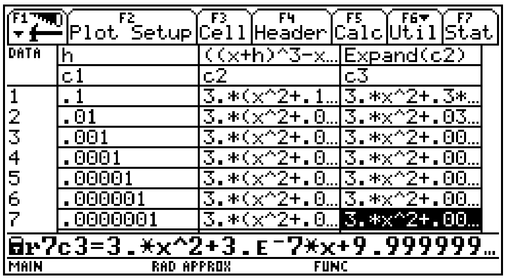


DATA	h	((x+h)^2-x…	Expand(c2)
	c1	c2	c3
1	.1	2.*(x+.05)	2.*x+.1
2	.01	2.*(x+.005)	2.*x+.01
3	.001	2.*(x+.000…	2.*x+.001
4	.0001	2.*(x+.000…	2.*x+.0001
5	.00001	2.*(x+.000…	2.*x+.00001
6	.000001	2.*(x+.000…	2.*x+.0000…
7	.0000001	2.*(x+.000…	2.*x+.0000…

Thus the result appears to be of the form *2x* plus a quantity which tends to zero.

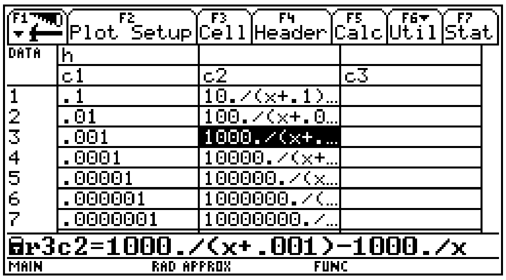
Editing column c2 allows us to easily repeat the process for other functions, for example $f(x) = x^3$

c2 = ((x + c1)^3 - x^3) / c1



DATA	h	((x+h)^3-x…	Expand(c2)
	c1	c2	c3
1	.1	3.*(x^2+.1…	3.*x^2+.3*…
2	.01	3.*(x^2+.0…	3.*x^2+.03…
3	.001	3.*(x^2+.0…	3.*x^2+.00…
4	.0001	3.*(x^2+.0…	3.*x^2+.00…
5	.00001	3.*(x^2+.0…	3.*x^2+.00…
6	.000001	3.*(x^2+.0…	3.*x^2+.00…
7	.0000001	3.*(x^2+.0…	3.*x^2+.00…

Fourth International Derive TI-89/92 Conference



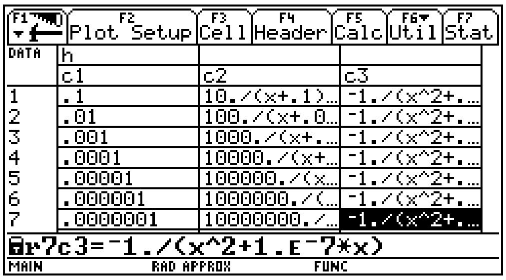
Unfortunately the limitations of the display do not make it obvious in this example that the terms after the $3x^2$ do indeed tend to zero.

Let us try another function: *f(x) = 1/x*

c2 = (1/(x + c1) - 1/x) / c1

Again, the output in column c2 is not obvious, but scrolling down the column and keeping an eye on the entry line, we see that we have the sum of two fractions. In such a case the function COMDENOM( ) comes in handy:

c3 = COMDENOM(c2)



and we see that the result simplifies to the form $\dfrac{-1}{x^2 + kx}$ where k tends to zero, giving the expected result.

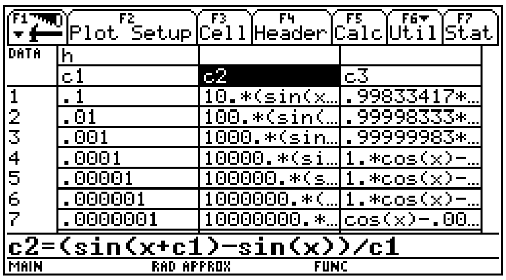
Finally, here is a very interesting example: *f(x) = sin(x)*

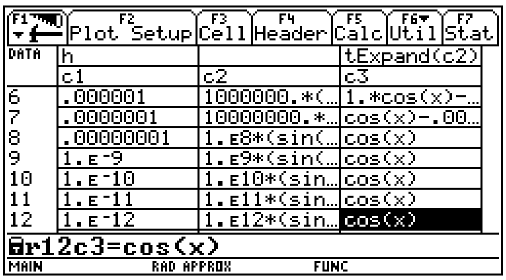
c2 = (sin(x+c1) - sin(x)) / c1

and to expand out the compound angle:

c3 = TEXPAND(c2)

The initial screen seems to indicate a result of the form *a.cos(x) + b.sin(x)* where a tends to 1 and b tends to 0. However, scrolling down a few lines gives a remarkable result:

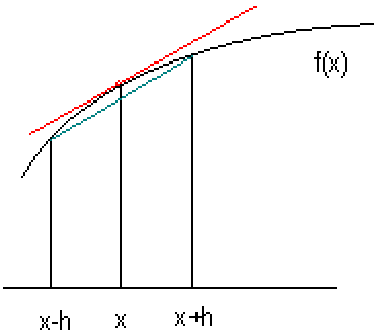




Contents

The above approach to "differentiation by first principles" is based on forward differences. This is the standard approach generally take in schools and colleges, since the algebra is not too difficult to do by hand. However, the central difference approach, namely

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$ is not only more quickly convergent but diagrammatically more intuitive and appealing:



We shall now investigate this method using a TI-92 spreadsheet.
Start a new datasheet called "diff2" in your "spread" folder.

We shall revert to the original example of $f(x) = x^2$ and compare the two methods:

c1 = SEQ(0.1^i , i , 1 , 10)
c2 = ((x + c1)^2 - x^2) / c1
c3 = ((x + c1)^2 - (x - c1)^2) / (2*c1)

F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	h	Forward	Central
	c1	c2	c3
1	.1	2.*(x+.05)	2.*x
2	.01	2.*(x+.005)	2.*x
3	.001	2.*(x+.000…	2.*x
4	.0001	2.*(x+.000…	2.*x
5	.00001	2.*(x+.000…	2.*x
6	.000001	2.*(x+.000…	2.*x
7	.0000001	2.*(x+.000…	2.*x

c3=((x+c1)^2-(x-c1)^2)/(2*c1)
MAIN RAD APPROX FUNC

Amazing! Well, perhaps not so amazing if we approached it algebraically pen-and-paper, since everything drops out neatly in the case of $f(x) = x^2$.

F1 | F2 Plot Setup | F3 Cell | F4 Header | F5 Calc | F6 Util | F7 Stat

DATA	Central	ErrorFwd	ErrorCnt
	c3	c4	c5
1	3.*(x^2+.0…	.3*x+.01	.01
2	3.*(x^2+.0…	.03*x+.0001	.0001
3	3.*(x^2+.0…	.003*x+.00…	.000001
4	3.*(x^2+3.…	.0003*x+1.…	1.ᴇ-8
5	3.*(x^2+3.…	.00003*x+1…	1.ᴇ-10
6	3.*(x^2+3.…	.000003*x+…	1.ᴇ-12
7	3.*(x^2+3.…	.0000003*x…	1.ᴇ-14

c5=c3-d(x^3,x)
MAIN RAD APPROX FUNC

Perhaps we could now compare the accuracy of the two methods by considering their errors. Take the case $f(x) = x^3$.

c1 = SEQ(0.1^i , i , 1 , 10)
c2 = ((x + c1)^3 - x^3) / c1

c3 = ((x + c1)^3 - (x - c1)^3) / (2*c1)
c4 = c2 - d(x^3,x)
c5 = c3 - d(x^3,x)

So we see that in this case, the error of the second method is superior, being independent of *x*. Also, in this particular example, we see most clearly confirmation of the well-known result that the error of this method is of order h^2.

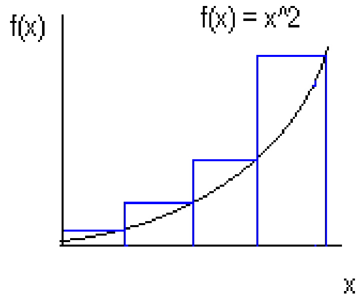
TASK: Use the spreadsheet approach to investigate other derivatives by first principles.

Activity 7: Integration and areas

In the previous activity we considered differentiation by first principles. We shall now consider integration and show how the integral is related to the area bounded by the graph. We shall do so on the basis of upper Riemann sums and use the test function $f(x) = x^2$.

We shall demonstrate how the area beneath the curve between the limits $x = 0$ and $x = t$ approaches $\frac{t^3}{3}$ as the number of rectangles increases.

Suppose there are *n* rectangles.
Each will have width *t/n*.
The height of the i'th rectangle will be $(i*(t/n))^2$ and hence the area will be $(t/n)*(i*(t/n))^2 = (t/n)^3 * (i)^2$

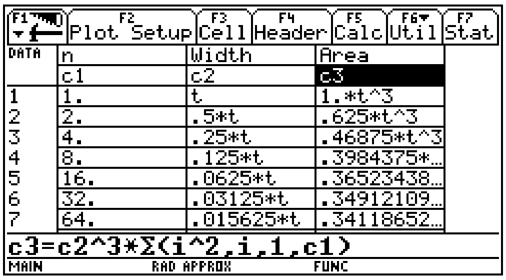


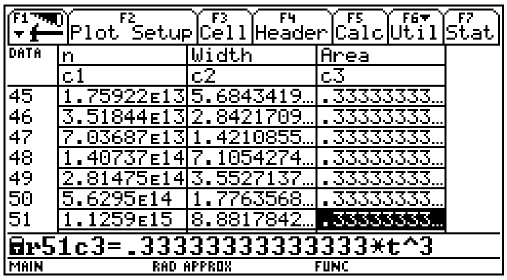
Therefore the total area of the rectangles is $\sum_{i=1}\left(\frac{t}{n}\right)(i)^2$ for some specified *n*.

This simplifies to $\left(\frac{t}{n}\right)^3 \sum_{i=1}^{n} (i)^2$ and we shall take *n* to be 1, 2, 4, 8, 16, ….

Start a new datasheet called "riemann" in your "spread" folder.

c1 = SEQ(2^i , i , 0 , 50)
c2 = t / c1
c3 = (c2^3) * Σ(i^2 , i , 1 , c1)





Note how the result converges, slowly but monotonically.

TASK 1: Verify some other integrals using the upper Riemann sum, by modifying this spreadsheet.
TASK 2: Try to implement other methods (midpoint rule, trapezoidal rule) and compare their convergence.

Activity 8: Reduction formulae

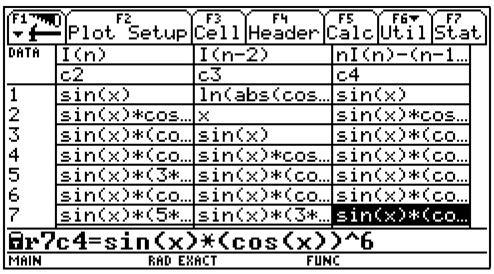
Sometimes it is possible to express an integral in terms of a related, but simpler, integral. This is known as a reduction formula. A common application is for high powers of trigonometric functions. For example,

$$\textit{If}\quad I_n = \int \cos^n x\,dx\ ,\ \textit{then}\ nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

A spreadsheet approach is suitable for investigating reduction formulae, since it is possible to list results for all values of n (n = 1, 2, 3, 4, …)

Firstly, let us verify the above reduction formula.
Start a new datasheet called "reduct" in your "spread" folder.

c1 = SEQ(i , i , 1 , 10)
c2 = ∫ ((cos(x))^c1 , x)
c3 = ∫ ((cos(x))^(c1-2) , x)
c4 = c1*c2 - (c1-1)*c3



Fourth International Derive TI-89/92 Conference

By scrolling down column c4, we see that it indeed appears to be the case that

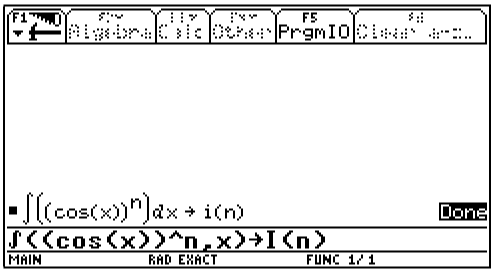
$$n I_n - (n-1) I_{n-2} = \sin x \cos^{n-1} x$$

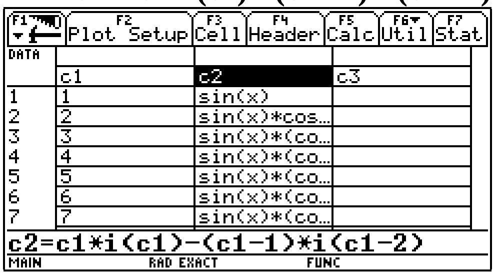
thus verifying the reduction formula.

It is always useful to remember that functions defined in the Home screen retain their definition in the Data/Matrix Editor. A more succinct implementation of the above is:

∫((cos(x))^n , x) STO> I(n)

c1 = SEQ(i , i , 1 , 10)
c2 = c1*I(c1) - (c1 - 1)*I(c1 - 2)





TASK 1: Verify the following reduction formulae:

$$\textit{If} \quad I_n = \int \tan^n x\,dx\,, \quad \textit{then} \quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

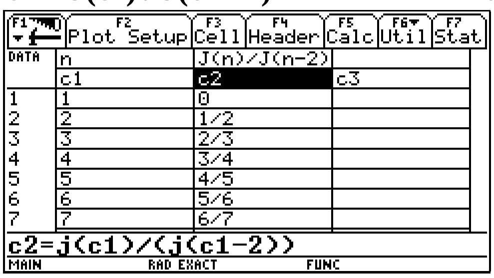
$$\textit{If} \quad I_n = \int \sin^n x\,dx\,, \quad \textit{then} \quad nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

A particular application of reduction formulae leads to Wallis's Formulae.

$$\textit{Let} \quad J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

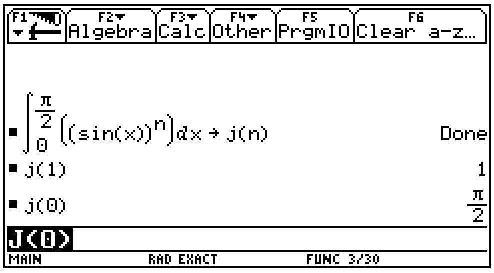
We shall demonstrate the relationship between J_n and J_{n-2}

In Home screen: **∫ ((sin(x))^n , x , 0 , π/2) STO> J(n)**
c1 = SEQ(i , i , 1 , 10)
c2 = J(c1) / J(c1 - 2) *ensure EXACT MODE*



Thus we find that $\dfrac{J_n}{J_{n-2}} = \dfrac{n-1}{n}$ *whence* $J_n = \dfrac{n-1}{n} J_{n-2}$

Thus, when n is odd, we can write J_n in terms of J_1, and when n is even we can write J_n in terms of J_0. The values of J_1 and J_0 can be quickly found from the Home screen:

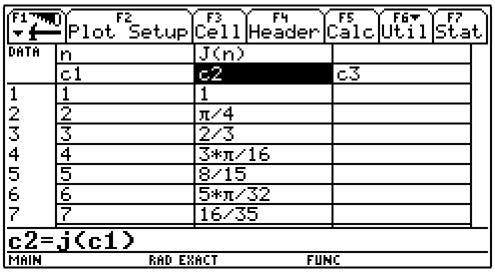


Thus with little effort we have deduced Wallis's Formula:

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{(n-1)(n-3)(n-5)\ldots(6)(4)(2)}{(n)(n-2)(n-4)\ldots(7)(5)(3)} \quad \textit{when n odd}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{(n-1)(n-3)(n-5)\ldots(5)(3)(1)}{(n)(n-2)(n-4)\ldots(6)(4)(2)} \times \frac{\pi}{2} \quad \textit{when n even}$$

and these results can be displayed in a spreadsheet:



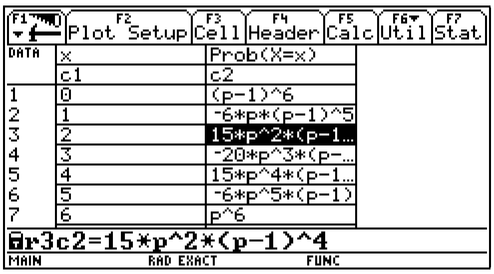
DATA	n	J(n)	
	c1	c2	c3
1	1	1	
2	2	π/4	
3	3	2/3	
4	4	3*π/16	
5	5	8/15	
6	6	5*π/32	
7	7	16/35	

c2=j(c1)

Activity 9: Probability distributions

A numerical spreadsheet is an ideal environment for presenting the probabilities given by a specified discrete probability density function, since $P(X = x)$ is given as a function of x. Furthermore, it is possible to change the values of the parameters and observe the change in the shape of the distribution.

Let us run through a Binomial distribution, and generate some distributions.
Start a new datasheet called "bindist" in your "spread" folder



DATA	x	Prob(X=x)
	c1	c2
1	0	(p-1)^6
2	1	⁻6*p*(p-1)^5
3	2	15*p^2*(p-1…
4	3	⁻20*p^3*(p-…
5	4	15*p^4*(p-1…
6	5	⁻6*p^5*(p-1)
7	6	p^6

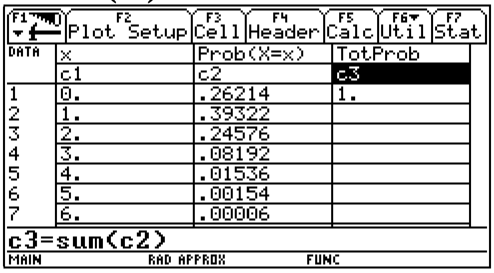
r3c2=15*p^2*(p-1)^4

We shall consider a random variable $X = Bin(n,p)$ with $n = 6$ and p variable.

Fourth International Derive TI-89/92 Conference

c1 = SEQ(i , i , 0 , 6)
c2 = (6!/(c1!*(6-c1)!))* p^c1 * (1-p)^(6-c1)

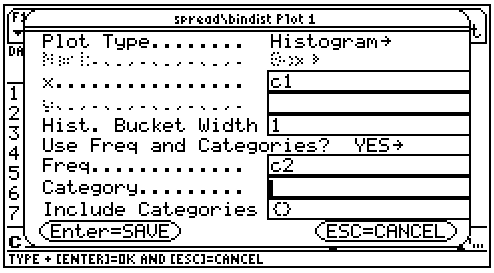
The value of *p* can be set (e.g. to 0.2) by entering **0.2 STO> p** in the Home screen.
By defining **c3 = sum(c2)** we can confirm that the total probability is 1.



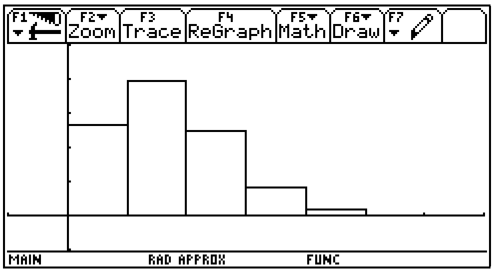
DATA	x	Prob(X=x)	TotProb
	c1	c2	c3
1	0.	.26214	1.
2	1.	.39322	
3	2.	.24576	
4	3.	.08192	
5	4.	.01536	
6	5.	.00154	
7	6.	.00006	

c3=sum(c2)

The graph of the distribution can be displayed.
F2: Plot Setup
F1: Define and fill in the dialog box as shown



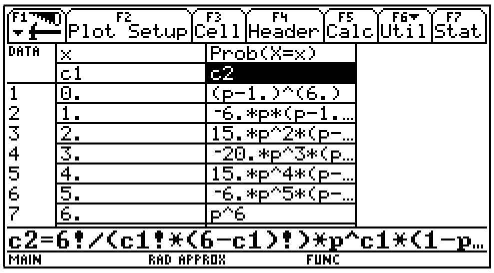
[♦] WINDOW and choose -1<x<7 , -0.1<y<0.5
[♦] GRAPH



The algebraic facilities of the TI-92 allow us to analyse the general case of the Binomial distribution. Let us now derive expressions for its mean and variance.

First delete the value of 0.2 we assigned to *p* earlier:
[2nd] VARLINK and scroll down until you find *p*
F1: Manage
1: Delete
We can also get rid of our column c3 (the total probability)
F6: Util

2: Delete
3: Column

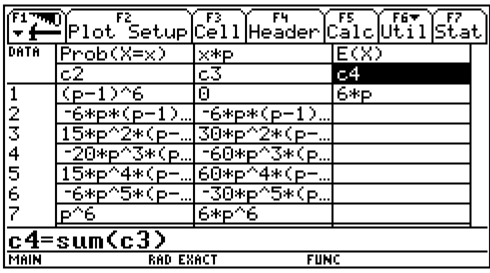


DATA	x	Prob(X=x)
	c1	c2
1	0.	(p-1.)^(6.)
2	1.	-6.*p*(p-1.…
3	2.	15.*p^2*(p-…
4	3.	-20.*p^3*(p…
5	4.	15.*p^4*(p-…
6	5.	-6.*p^5*(p-…
7	6.	p^6

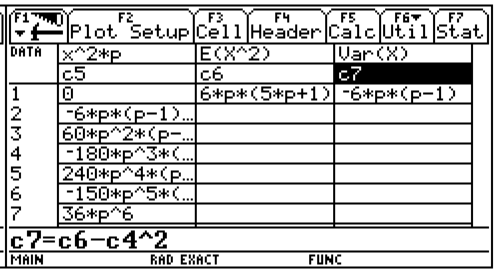
Recall that the mean of a probability distribution is given by $\sum x\, p$

and the variance is given by $\sum x^2 p - \left(\sum x\, p\right)^2$

c3 = c1 * c2
c4 = SUM(c4)
c5 = c1^2 * c2
c6 = SUM(c5)
c7 = c6 - c4^2



DATA	Prob(X=x)	x*p	E(X)
	c2	c3	c4
1	(p-1)^6	0	6*p
2	-6*p*(p-1)…	-6*p*(p-1)…	
3	15*p^2*(p-…	30*p^2*(p-…	
4	-20*p^3*(p…	-60*p^3*(p…	
5	15*p^4*(p-…	60*p^4*(p-…	
6	-6*p^5*(p-…	-30*p^5*(p…	
7	p^6	6*p^6	



DATA	x^2*p	E(X^2)	Var(X)
	c5	c6	c7
1	0	6*p*(5*p+1)	-6*p*(p-1)
2	-6*p*(p-1)…		
3	60*p^2*(p-…		
4	-180*p^3*(…		
5	240*p^4*(p…		
6	-150*p^5*(…		
7	36*p^6		

Thus we achieve the well-known results $E(X) = np$, $Var(X) = np(1-p)$

In our example above, $n = 6$. However, to verify the results for other values of n, all we need to do is to edit the definition of column c1 and choose a new upper limit:

c1 = SEQ(i , i , 0 , *new n* **)**

as well as editing the definition of column c2 to replace the 6 by whatever new value we choose.

c2 = (*new n***!/(c1!*(** *new n* **-c1)!))* p^c1 * (1-p)^(** *new n* **-c1)**

Try this!

Fourth International Derive TI-89/92 Conference

Activity 10: Miscellaneous tasks for further investigation

(Please fill in the blank boxes with any ideas that occur to you during the workshop!)

By setting up two columns of randomly generated complex numbers, verify by example the following rules of complex numbers:

$mod(z_1 z_2) = mod(z_1)mod(z_2)$ $\quad$ $mod(z_1/z_2) = mod(z_1)/mod(z_2)$

$arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $\quad$ $arg(z_1/z_2) = arg(z_1) - arg(z_2)$

Can de Moivre's Law be demonstrated in a similar fashion?

Investigate the approximation for the second derivative:

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \qquad \text{as } h \text{ tends to zero}$$

Verify the small angle approximations

$\sin\theta \cong \theta$ $\quad$ $\cos\theta \cong 1 - \theta^2/2$

For what range of θ are they within 1%?

What degree of improvement is obtained by taking the next terms in the Taylor expansions, namely:

$\sin\theta \cong \theta - \theta^3/6$ $\quad$ $\cos\theta \cong 1 - \theta^2/2 + \theta^4/24$

Discover the product rule for differentiation by setting up a column (list) of functions of the form $x^n \sin x$ for $n = 1,2,3,4,\ldots$ and differentiating this list.

Discover the quotient rule.

Discover the chain rule.

Investigate the incremental increase rule $\delta A \approx \dfrac{dA}{dt}\delta t$