Fourth International Derive TI-89/92 Conference
Liverpool John Moores University, July 12-15, 2000

Locating Multiple Roots of Polynomials

Steven Schonefeld
Tri-State University, USA
e-mail: schonefeld@fwi.com

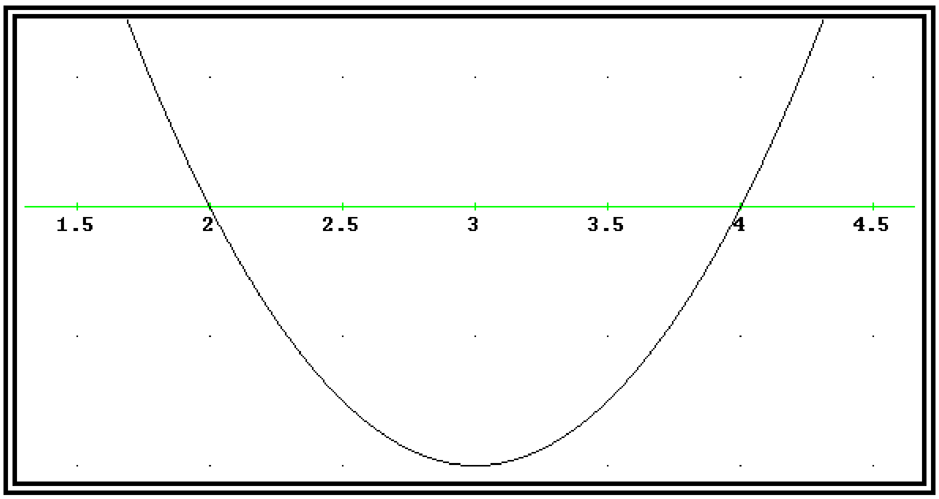
Let F(x) be a polynomial. We say x = r is a ***root*** (or ***zero***) of F(x) if F(r) = 0. In this case, the Fundamental Theorem of Algebra assures us that there exists a polynomial, Q(x), such that

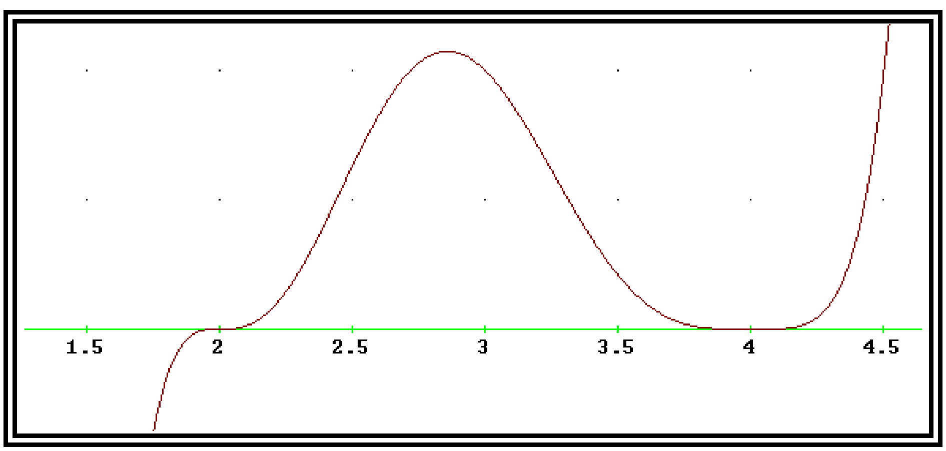
$$F(x) = (x - r)Q(x). \qquad (1)$$

We say that x = r is a ***simple root*** if equation (1) is true and Q(r) is non-zero.
We say that x = r is a ***multiple root of order k*** (for F(x)) if

$$F(x) = (x - r)Q(x) \qquad (2)$$

with Q(r) non-zero. In figure 1 opposite, we see the graph of a polynomial having simple roots at x = 2 and x = 4. As illustrated by this figure, the derivative of a polynomial is non zero at each simple root. We can visually locate these simple roots with relative ease and accuracy.





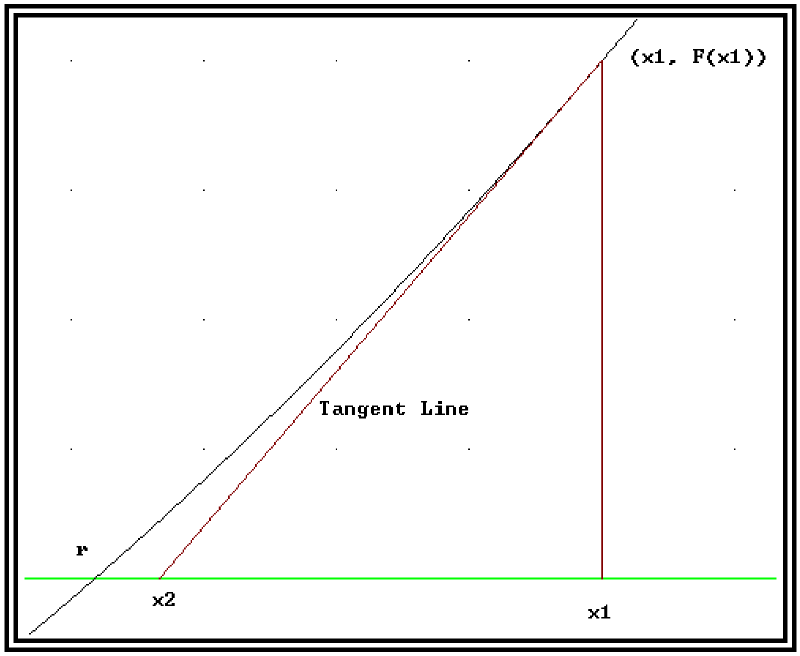
In figure 2 opposite, we see the graph of a polynomial having multiple roots at x = 2 and x = 4. As illustrated by this figure, the derivative of a polynomial is zero at each multiple root. Near a multiple root, the graph of F(x) is virtually indistinguishable from the x-axis. This makes it difficult to locate the roots.

Newton's Method.

For polynomials of degree greater than four, an approximation method must be used to locate the roots. That is, we may never find the exact roots, but we may approximate each root (say, to 15-digits accuracy.) One favorite method of locating roots is Newton's Method. With this method, we start with a number, x_1, (called an ***initial guess***) near the desired root, r. From this we (perform an ***iteration*** to) get x_2; from x_2 we get x_3; etc. For an initial guess reasonably near r, Newton's method should give us a sequence

$$\{x_1, x_2, \ldots, x_n\}$$

of approximations that will converge to x = r. Geometrically, x_2 is found from x_1 by constructing the tangent line at the point $(x_1, F(x_1))$. Then, x_2 is the x-coordinate of the point where this tangent line intersects the x-axis – as shown in Figure 3 above. The formula for determining x_{n+1} from x_n is:



$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \tag{3}$$

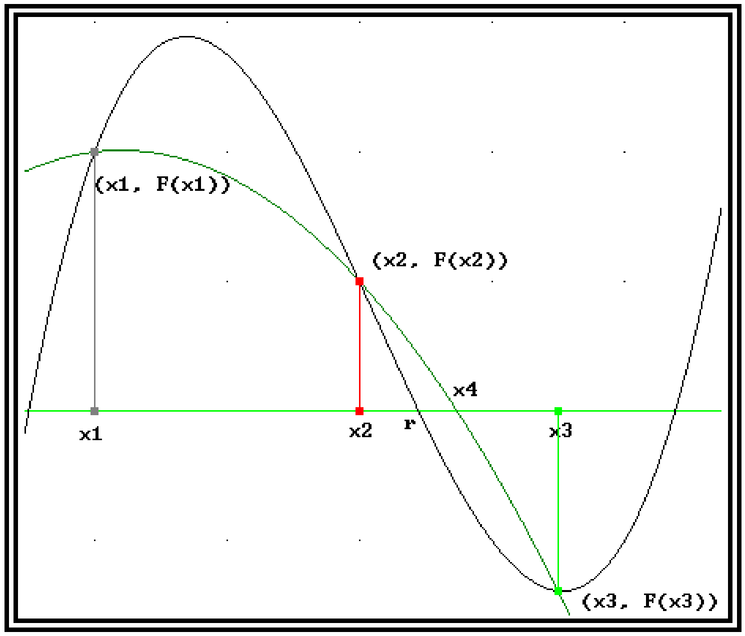
where $F'(x)$ is the first derivative of $F(x)$. Iterations continue until two successive approximations are within 10^{-15} or a maximum number of iterations is reached. Newton's method works quite well for locating simple roots of polynomials – provided that we can find a suitable initial guess. It is not so successful in locating a multiple root.

Müller's Method.

Perhaps the favorite method for finding the roots of a polynomial is Müller's Method. We will give the steps involved in one iteration. We start with three initial guesses: x_1, x_2, x_3. We find the quadratic polynomial, q(x), which passes through the three points:

$$(x_1, F(x_1)), (x_2, F(x_2)), (x_3, F(x_3)).$$

We expect q(x) to have two roots. We take x_4 to be the root, which is closer to x_4. See figure 4 opposite for a typical first iteration of Müller's Method. This same process is applied to x_2, x_3, x_4 to give x_5, etc. As with Newton's Method, the iterations continue until the last two values of x are within 10^{-15} or a maximum number of iterations is reached.



Once one root, r_1, has been found, the function $F(x)$ is re-defined as the original function divided by $(x - r_1)$. This process is called ***deflation***. Applying Müller's Method to this deflated F(x) should cause the next sequence of iterations to converge to a root different from r_1 (at least if r_1 is a simple root.) After a second root, r_2, is found, the function $F(x)$ is again deflated so that it is the original function divided by $(x - r_1)(x - r_2)$. This process continues until all the roots are found. If n is the degree of the polynomial and all the roots are simple, it is easy to tell when we have found all the roots.

Müller's Method is so robust that any three initial guesses will lead to the location of a root – even if all the roots of $F(x)$ are complex. Newton's Method will also locate complex roots. Of course, Newton's method requires an initial guess fairly close to the root. Near a multiple root, a straightforward application of Müller's Method will require too many iterations to be practical. So, the number of iterations must be limited, say to 30. See [1], for more information about Newton's Method and Müller's Method.

Simple Roots from Multiple Roots.

Suppose that $F(x)$ has a multiple root, r, of order k. That is, $F(x)$ may be expressed as in equation (2). We divide $F(x)$ by its derivative to get the following:

$$\frac{F(x)}{F'(x)}=\frac{(x-r)^k\,Q(x)}{k(x-r)^{k-1}Q(x)+(x-r)^k\,Q'(x)}=\frac{(x-r)Q(x)}{kQ(x)+(x-r)Q'(x)}. \qquad (4)$$

Near x = r, Q(x) is non-zero. So we may divide numerator and denominator of the last expression in (4) by Q(x) to obtain

$$\frac{F(x)}{F'(x)}=\frac{(x-r)}{k+\dfrac{(x-r)Q'(x)}{Q(x)}}\approx\frac{1}{k}(x-r).$$

That is, for x near, but not equal to r, the function

$$\Phi(x)=\frac{F(x)}{F'(x)} \qquad (5)$$

behaves as though it has a simple root x = r. Hence, Newton’s Method can be applied to Φ(x) to determine the roots of F(x). For each root, all we need is a reasonably good initial guess. This initial guess is obtained by applying Müller’s Method to F(x).

Determining Root Order.

For x near the root r of F(x), we have

$$\Phi(x)=\frac{F(x)}{F'(x)}\approx\frac{1}{k}(x-r). \qquad (6)$$

Thus, for x near r, we have

$$\Phi'(x)\approx\frac{1}{k}.$$

Using a central difference approximation to the derivative of Φ, we get

$$\Phi'(x)\approx\frac{\Phi(x+\delta)-\Phi(x-\delta)}{2\delta}\approx\frac{1}{k}$$

for reasonably small (non-zero) values of δ. Taking reciprocals, gives

$$k\approx\frac{2\delta}{\Phi(x+\delta)-\Phi(x-\delta)} \qquad (7)$$

for x near r and δ reasonably small. In particular, suppose that we have obtained (with Newton’s Method) an approximate root, r_A, so that $|r_A-r|<10^{-15}$. We may take $\delta = 8\cdot 10^{-15}$ and $x = r_A$ to assure that Φ(x + δ) and Φ(x – δ) are defined and not equal – yet (7) will give a good approximation to k. Since k is a natural number, the exact value of k is given by the

DERIVE function

$$FLOOR\left(RE\left(\frac{2\delta}{\Phi(r_A+\delta)-\Phi(r_A-\delta)} \right) + \frac{1}{2} \right)$$

for r_A and δ as above.

We need to know the order of each root of polynomial F(x) for two reasons. The first use of root order is in the application of Müller's Method. Suppose that we have found a root, r_1, and it has order k_1. Then, for the deflation step, we divide by $(x-r_1)^{k_1}$ to prevent Müller's Method from returning the root r_1 a second time. The second, and perhaps more important use of root order, is to tell when we have found **all** the roots of F(x). Simply counting the number of roots is not sufficient to ascertain that we have found all the roots. Suppose the polynomial F(x) has degree n and we have found roots $\{r_1, r_2, \cdots, r_m\}$ having respective root orders $\{k_1, k_2, \cdots, k_m\}$, we will have found all the roots when $k_1 + k_2 + \cdots + k_m = n$. That is, we can quit looking for roots when the sum of the root orders equals the degree of polynomial F(x). In this way, we may locate all the roots of a polynomial having multiple roots.

References

[1] Schonefeld, Steven, ***Numerical Analysis via Derive***, MathWare, Urbana, IL, 1994.