Contents

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MATHEMATICS, MELODY AND BARBERSHOP HARMONY

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Harmony

A basis for musical harmony is provided by Fourier's theorem which treats temporally periodic phenomenon f(t) of period T, as a superposition of simple harmonic oscillations of successively higher frequencies called harmonics, the successive amplitude A_n of which in contrast, decay away, as described by the Fourier series

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(n\omega t + \varepsilon_n\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right), \qquad ...(1)$$

where ω is the angular or radian frequency of a fundamental simple harmonic oscillation, which in terms of the secular frequency $\nu = 1/T$ (measured in Hz), is given by $2\pi\nu$, whilst t represents time in seconds.

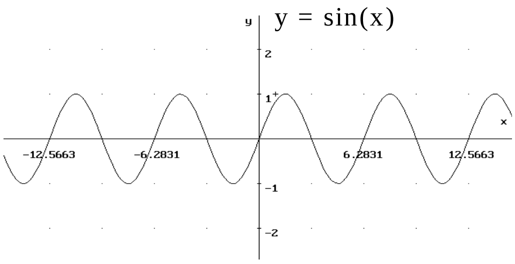
Equation (1) can be expressed in the alternative form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t), \qquad ...(2)$$

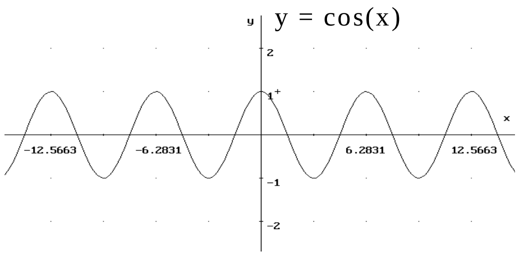
where the constants in equ.(2) are related to those in equ.(1) by $a_n = A_n\cos(\varepsilon)$ and $b_n = -A_n \sin(\varepsilon)$, whilst $\varepsilon_n = -\tan^{-1}\left(b_n / a_n\right)$ is the epoch of the n[th] harmonic. Furthermore the coefficients a_n and b_n can be calculated directly with the Euler formulae as

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt \text{ and } b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt .$$

According to equ.(1), the basic building blocks in Fourier series are the sine and cosine waves, depicted respectively in Figs. 1(a) and (b), where x is short ωt.



(a)



(b)

Fig. 1

Confining our attention to the sine wave, on the understanding that the same basic treatments will apply to the cosine wave, we graph, in Fig. 2, the corresponding sine waves of double, triple, quadruple, etc., the frequency of the sine wave in Fig. 1a.

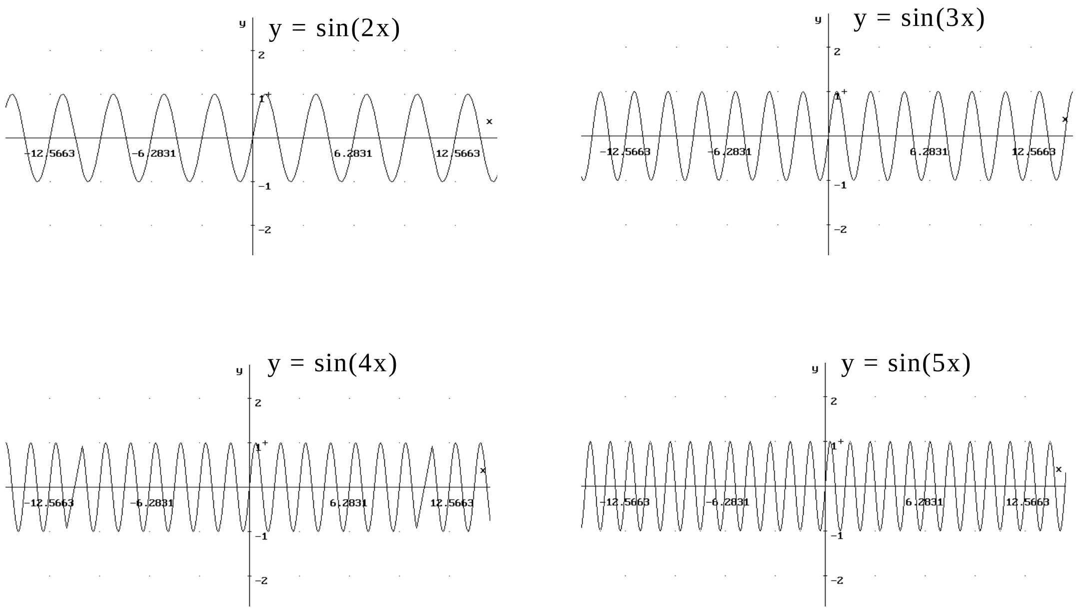
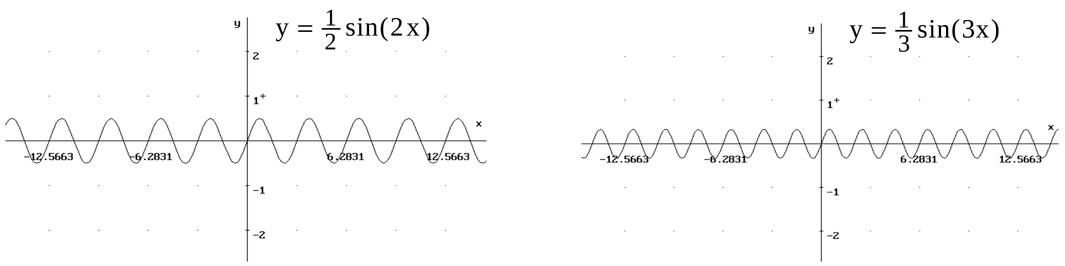


Fig. 2

These latter sine waves of multiple frequency constitute the sub-building blocks or harmonics of a Fourier series.

However, whilst it is mandatory that the frequencies increase in simple whole number multiples of the fundamental frequency, we are free to adjust the amplitudes of the successively higher frequency waves in any way we desire.

For example, we might take the original sine wave at unit amplitude, the sine wave of doubled frequency at half this amplitude, the tripled frequency sine wave at one third of ..., and so on. The graphs of these reduced amplitude waves, are shown in Fig. 3.



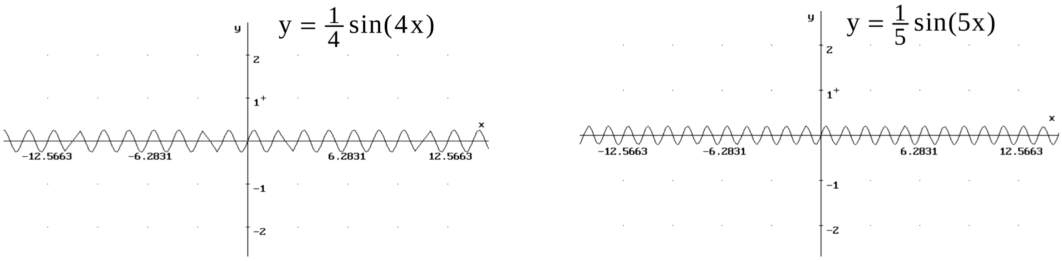


Fig. 3

The final step, is to superpose these waves, which is simply done by arithmetically adding them together.

To see the way the composite wave develops, we graph in turn in Fig. 4, the cumulative sums

(i) $\sin(x)+\frac{1}{2}\sin(2x)$,

(ii) $\sin(x)+\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)$,

(iii) $\sin(x)+\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)+\frac{1}{4}\sin(4x)$, etc.

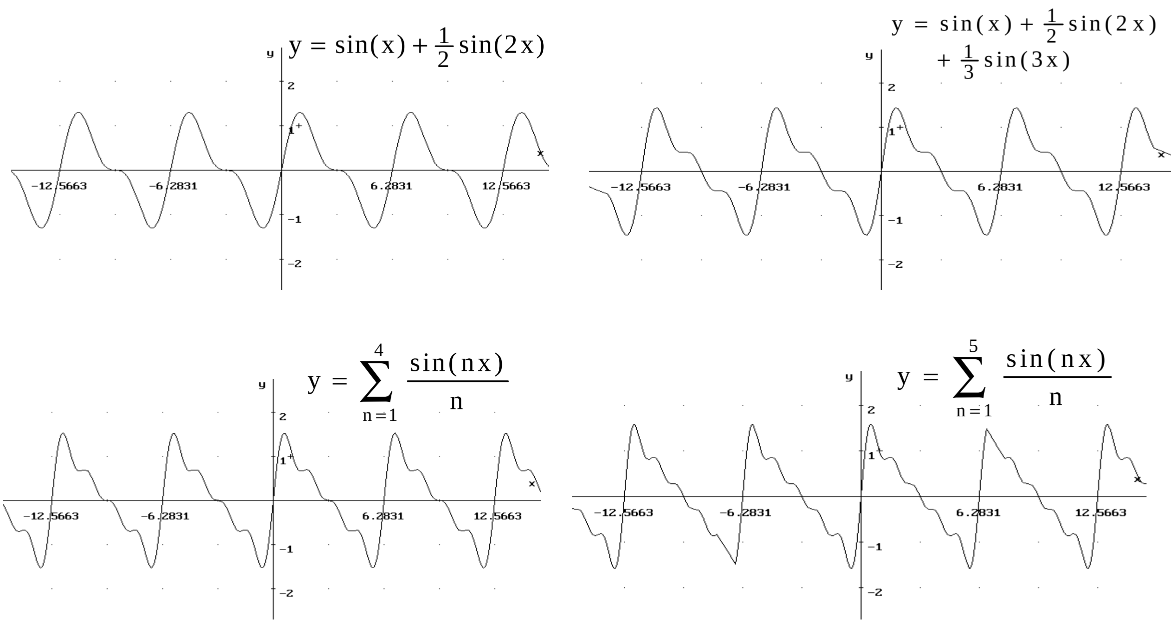


Fig. 4

If in the above sum, the argument x is returned to its original form ωt, then when ν = 66 Hz , the lowest component in the Fourier series would be a simple vibration heard roughly as bottom C in the bass clef as illustrated in Fig. 5. Furthermore the various sums in Fig. 4 would be heard as musical chords. The chord for the final entry in Fig. 4 is indicated in Fig. 5.

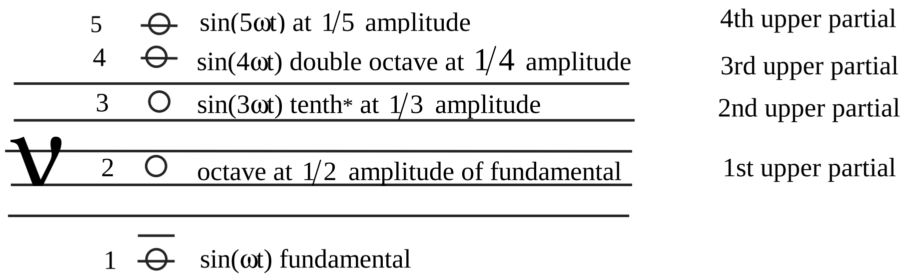


Fig. 5

The various tones in the above scheme are variously called harmonics or upper partials, the lowest tone being dubbed the ‘fundamental’.

Returning to the superposition problem, we show finally in Fig. 6, the graph of the first hundred terms, namely that of $y = \sum_{n=1}^{100} \frac{\sin(nx)}{n}$, of the series under consideration, where we note that the original wavy character has all but disappeared, to be replaced by repeating straight line segments, with a slight blip at points of discontinuity, which is a manifestation of the so-called Gibb’s phenomenon.

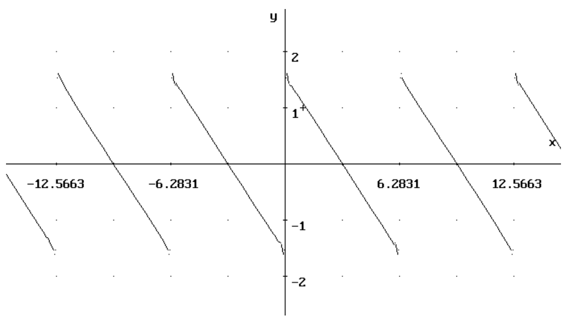


Fig. 6

The Gibb’s phenomenon persists even in the case of the sum of an infinitude of harmonics which would look like:

$$y = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \quad . \qquad ...(3)$$

A similar expansion pertains to the sawtooth wave defined more realistically by $f_R(t) = 2\alpha t/T \ : \ |t| < T/2$, periodic of period T, with α a constant, which is illustrated in Fig. 7.

Fourth International Derive TI-89/92 Conference

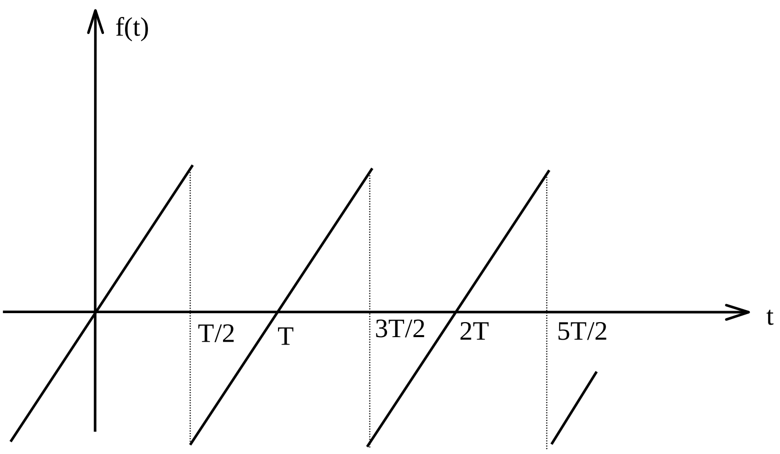


Fig. 7

The Fourier series of this wave is given by

$$f_R(t) \sim \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\omega t)\,, \qquad \text{...(4)}$$

where again ω = 2π/T is the angular frequency of the fundamental wave, defined as before in terms of the secular frequency ν = 1/T.

Apart from the additional parameters introduced here to give the waveform a more physical nature together with an algebraic sign, the only essential difference between the waves in Figs. 6 and 7 is that the phases of their respective harmonics differ by successive whole number multiples of half a period. However the human ear is apparently insensitive to such phase differences and so the above wave would sound the same as that described by

$$f_L = \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\omega t)}{n}\,, \qquad \text{...(5)}$$

which differs from y in equ.(3) essentially only by a multiplicative constant.

The amplitude magnitude versus frequency behaviour* of a given wave, which in fact defines its timbre, can be summarised by a bar chart of the type indicated in Fig. 8, whilst Fig. 9 attempts a musical realisation of this chart, when $\nu = 1/T = 66\ Hertz$.

*More realistically the quantity **20log$_{10}$|amplitude|** might be plotted against frequency.

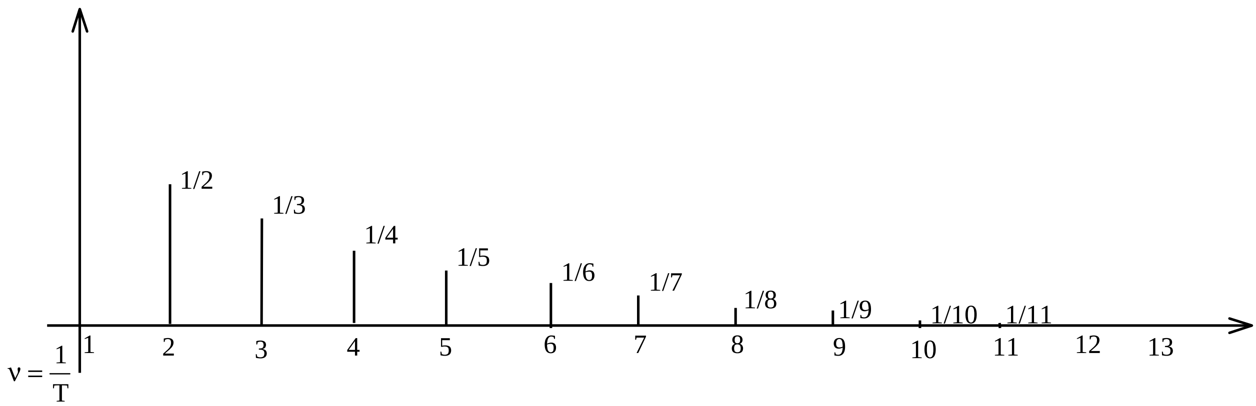


Fig. 8

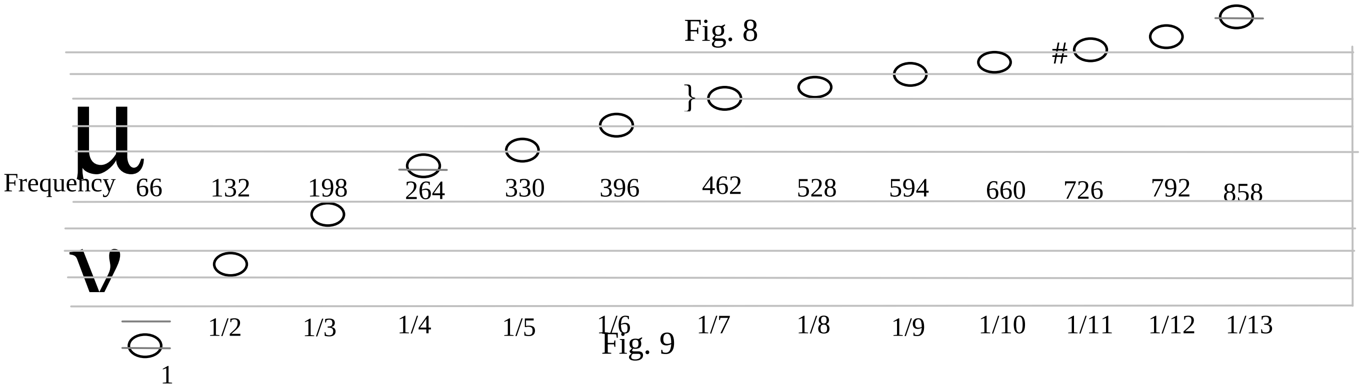


Fig. 9

It is important to note in the above scheme that the eleventh harmonic is sharper than we might otherwise expect [1,2]. Brass players can test this in an appropriate key, by ‘lipping’ the harmonics usually up to the eleventh, on a fixed valve position (trumpet or cornet etc) or a fixed slide position (trombone) assuming they start on a low enough note.*

It is left as an exercise for the reader to construct corresponding schemes for the square and triangular waves, defined as the periodic continuations of the respective origin centred representatives:

(i) $$f_9(t) = \begin{cases} \alpha & : \ |t| < T/2 \\ 0 & : \ |t| > T/2 \end{cases},$$

(ii) $$f_\Delta(t) = \alpha \begin{cases} (1+2t/T) \\ 2t/T \\ 1-2t/T \end{cases},$$

for which the corresponding Fourier series are respectively

(iii) $$f_9(t) = \frac{4\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\cos\left((2n+1)\omega t\right)}{2n+1}$$

and

(iv) $$f_\Delta(t) = \frac{8\alpha}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \sin\left((2n+1)\omega t\right)}{(2n+1)^2}.$$

These series indicate that the even harmonics in the above waves are missing, which imparts a rather reedy sound to the resulting timbres.

*Bottom F# on the trumpet (i.e. concert E).

Composite waveforms

According to the above, though the oscillatory motions of a vibrating object may be quite complex, it turns out that these oscillations may be analysed into a sum of simple harmonic oscillations, the pattern of which, as mentioned above, determines the timbre of the sound it emits.

Thus when a guitar string is plucked, the motion of the entire string is made up of a whole set of simpler vibrations. The maximum distance the string deviates from its equilibrium position, depending on how hard the string is plucked, determines the amplitude of the vibration.

If the simple back-and-forth motion of the vibrating source were the only phenomenon involved in creating sound, then all vibrating components of a given kind - like those in musical instruments - would probably sound much the same. However it turns out in the case of the guitar for instance, that a given string not only vibrates at its entire length, but also at one-half its length, one-third, one-quarter, one-fifth, and so on. These additional vibrations respectively occur at twice, three times, four times etc. the frequency of the full string length vibration, but with generally weaker amplitudes. Importantly each of these sub-vibrations, including the fundamental one, can be described by sinusoidal functions of time.

Thus the displacement of a guitar string with time can in fact be viewed as a superposition of sinusoidal waves of successive whole number frequencies, the amplitudes of which however, due to friction, decay exponentially with time, until that is, the string is plucked again.

For a plucked string of length ℓ, fixed at both ends with initial profile:

$$u(x,0) = \begin{cases} bx/\xi & : \; 0 \le x \le \xi \\ b\left(\dfrac{x-\ell}{\xi-\ell}\right) & : \; \xi \le x \le \ell \end{cases}, \qquad \text{...(6)}$$

illustrated in Fig. 10,

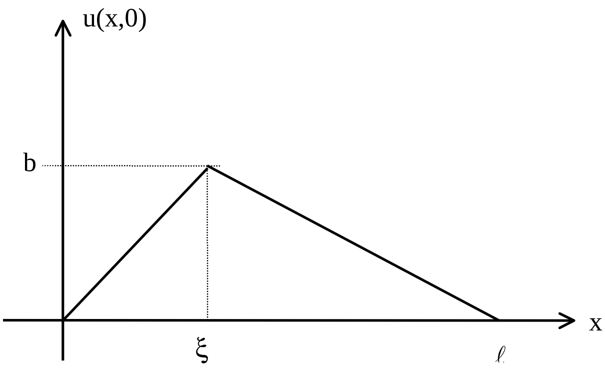


Fig. 10

a simple model for the subsequent displacement u(x, t) of the string at station x and time t, is provided by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{...(7)}$$

where $c = \sqrt{T/\sigma}$ represents the phase speed in terms of the tension T in the string and the line density σ of the string.

The solution to the above equation, which is

$$u(x,t) = \frac{2b\ell^2}{\pi^2 \xi(\ell-\xi)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi\xi}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell}\right) \qquad \text{...(8)}$$

demonstrates that in the case of a plucked string

(i) the harmonics fall as $1/n^2$,

(ii) the amplitudes of the composite tones increase as the plucked point approaches either end of the string - in the case of the guitar this would preferably be the bridge end,

(iii) any particular harmonic vanishes at a node, where $\sin\left(\frac{n\pi\xi}{\ell}\right) = 0$.

In the case of the piano, the strings are struck with a hammer of thickness ε imparting an initial velocity to the string, a profile which is given by

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0 & : \quad 0 \le x \le \xi - \varepsilon \\ v & : \quad \xi - \varepsilon \le x \le \xi - \varepsilon\,. \\ 0 & : \quad \xi - \varepsilon \le \ell \end{cases} \qquad \text{...(9)}$$

For this case the solution of the wave equ.(7) becomes

$$u(x,t) = \frac{4v\ell}{c\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi\varepsilon}{\ell}\right) \sin\left(\frac{n\pi\xi}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi ct}{\ell}\right). \qquad \text{...(10)}$$

For a thin hammer this can be modified to

$$u(x,t) = \frac{4c\varepsilon}{c\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi\xi}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi ct}{\ell}\right). \qquad \text{...(11)}$$

Comparing this solution with that of the plucked string, we see that the upper partials are stronger in amplitude, falling off as they do in proportion to $1/n$ instead of $1/n^2$.

The fundamental angular frequency of the above waves is ω = σc/ℓ, whilst the frequency of the gravest mode, occurring for n = 1, is $\nu = c/2\ell = \frac{1}{2\ell}\sqrt{\frac{T}{\sigma}}$, which according to Mersenne's laws,* corresponds to the note we think we hear.

Waves on strings like those the above, can be observed by switching on a television in an otherwise darkened room and holding a guitar in such a way as to view the guitar's strings

*The period of vibration would be $2\ell\sqrt{\sigma/T}$.

with the television screen in the background. The television screen acts as a stroboscope, picking out low alias vibrations of a plucked string which can be seen as visible waves, superimposed on the vibrating guitar string.

It is important to note however that the waves do not travel along the string, which may be the impression conveyed by the above stroboscopic test, since as indicated by equ.(8), the sections of the string oscillate as standing waves. This means that at certain equi-spaced points along the string, the motion is stationary. At such a point the corresponding harmonic may be sounded by simply touching the string, rather than fully stopping it with the full pressure of the player's fingers.

String players use this effect to play harmonics and furthermore to tune their instruments.

Thus a double bass player can, with the bow sound on the G-string, the octave of the D-string, by touching with the little finger the 1/3 node on the G-string, whilst at the same time sounding the same note on the D-string by touching its quarter-way node with the index finger. Drawing the bow across the two strings so constrained simultaneously, sounds both notes together, whence they can be fine tuned relative to one another. This process can be repeated to tune the remaining strings.

Given that the strings of the double bass are tuned in fourths from bottom E upwards, it is left as an exercise for the reader to figure how cellists might correspondingly tune their instruments given that the strings of the cello are tuned in fifths starting from bottom C; likewise for the violin whose strings are tuned in fifths starting from bottom g.

The oscillation of the vocal folds in voice production is governed essentially by an aerodynamic phenomenon in which the vocal folds are first sucked together - according to the Bernoulli effect - by air escaping from the trachea - after which they are blown apart by the accumulated air pressure underneath - depending on the muscular effort [3,21]. The resultant waveforms describing either aerial velocity or glottal area can be successfully modelled by a triangular wave which is a periodic continuation of the origin centred representative plotted in Fig. 11, of the form $g(t)=\begin{cases}1-|t|/\alpha & : |t|\le\alpha \\ 0 & : \alpha\le|t|\le T/2\end{cases}$, where T is the period of vibration of the vocal chords, whilst α is the fraction of a period for which the glottis is open during a given cycle of the vibration. We might thus call α the opening quotient.

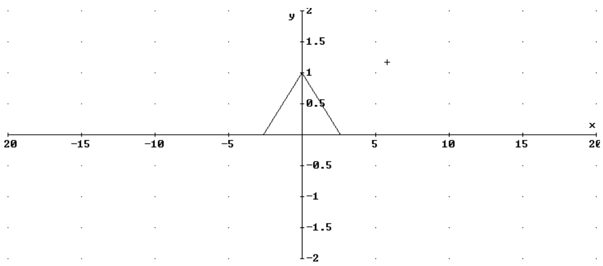
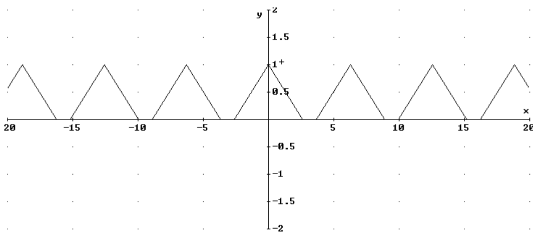
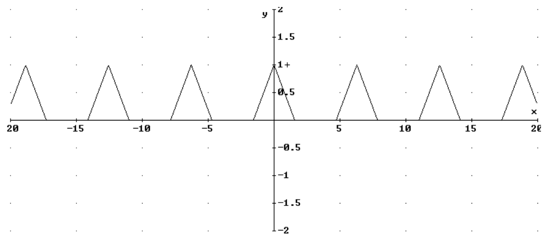


Fig. 11

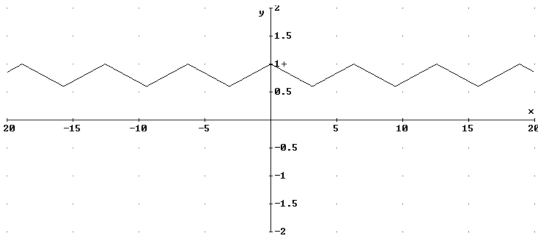
Plots of the periodic extensions of the above function for, a hypothetical period of T = 2π (although realistic vocal chord periods would be typically hundreds of times shorter than this), catering for the cases in which (i) $\alpha = 5\pi/6$, (iii) $\alpha = \pi/2$ and (iii) $\alpha = 5\pi/2$ are given in Fig. 12.



α = 5π/6



α = π/2



α = 5π/2 (falsetto effect)

Fig. 12

Given that the periodic extension of g(t), is periodic with period T = 2π, its Fourier expansion is given by

$$g(t) = \frac{1}{\pi}\left[\frac{\alpha}{2} + \frac{2}{a}\sum_{r=1}^{\infty}\left(\frac{1-\cos(r\alpha)}{r^2}\right)\cos(rt)\right]. \qquad \text{...(12)}$$

It can be seen from the above graphs that a decrease in α makes the graphs more pointed which strengthens the upper partials, so that more energy goes into the higher frequencies with a consequent increase in power output. Practised singers tend to have stronger upper partials in their vocal timbres.

Clearly in the production of low notes by the above effect the vocal cords are more relaxed than in the production of higher notes, when the vocal cords would be expected to be under greater tension. The larynx does in fact rotate upwards when higher notes are sung, as can be sensed by gently placing the index finger on the Adam's apple when singing an octave.

If we modelled the vocal frequency on the formula $\nu = \frac{1}{2\pi}\sqrt{\frac{T}{\sigma}}$, we might expect a quadrupling in the tension of the vocal cords in elevating a sung note by an octave.

Further increase in frequency can be produced by adducting the vocal cord - the so-called break - or perhaps by decreasing their mass σ per unit length. Finally a falsetto effect is produced by tensing the vocal cords to such an extent that they do not entirely close during a given cycle, which effect may be mimicked in equ.(12) by increasing α until the triangular waves overlap as illustrated in Fig. 12.

Formants

The composite waveform due to a vibrating component still does not entirely account for all the characteristics of sound produced by different instruments. One further attribute is determined more by the nature of the resonator to which the vibrating component is attached. The resonator in fact enhances some of the sinusoidal harmonics describing the vibrations, whilst attenuating others [21].

In the case of articulation, the resonator is the vocal tract as schematised in Fig. 13, whilst the resonances it produces manifest themselves as vowel sounds.

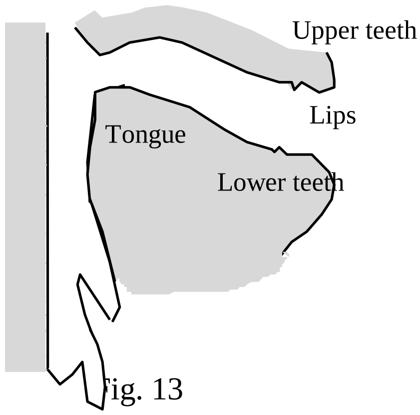


Fig. 13

The dynamics of a single resonator's behaviour may be described by the initial value problem

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_n^2 x=\omega_n^2 f(t)\ :\ x(0)=\dot{x}(0)=0\,, \qquad ...(13)$$

where γ and ω_n are constants, respectively representing a damping factor and the natural angular frequency of the resonator, whilst $\omega_n^2 f(t)$ is a forcing function.

Equ. (13) is in fact a canonical representation of the dynamical equation

$$\frac{\rho\ell}{A}\frac{d^2X}{dt^2}+\frac{\rho ck^2}{2\pi}\frac{dX}{dt}+\frac{\rho c^2}{V}X=F(t)\,, \qquad ....(14)$$

which describes the volume displacement of a plug of air, of density ρ, in the neck of length ℓ and sectional area A, of the idealised Helmholtz resonator of volume V, forced by the function F(t), shown in Fig. 14, which is perhaps the simplest model with which to start a formant theory. Further parameters in equ.(14) are the speed of sound in air, which on the adiabatic hypothesis is given by $c=\sqrt{\gamma p_0/\rho_0}$, in terms of the equilibrium pressure and density p_0 and ρ_0 respectively of the air and γ is its ratio of specific heats at constant temperature and volume. The quantity k is also included to account for frictional or dissipative effects [4].

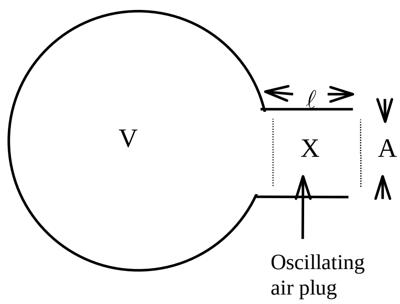


Fig. 14

A graph of the resonator's amplitude attenuation and amplification factors versus frequency as described by equ.(13) say, is called the envelope, or formant pattern, of the resonator. These characteristics may be obtained from the transfer function T(s) of the resonator, which is the Laplace transform of its impulse response $h(t) = \frac{\omega_n^2}{\omega_d} e^{-\gamma t/2} \sin(\omega_d t)$, where $\omega_d = \sqrt{\omega_n^2 - \gamma^2/4}$ is a damped angular frequency, which indicates that damped oscillations are flatter than natural ones. This latter function h(t) is in the simplest case, the solution of the differential equation (13) with its right-hand-side term replaced by Dirac's delta function δ(t)

Alternatively introducing the hypothetical solution $x(t) = T(\omega)e^{j\omega t}$, where $j = \sqrt{-1}$ is the imaginary unit, into equ.(13), with f(t) replaced by $e^{j\omega t}$, leads to the frequency response $T(\omega) = \frac{\omega_n^2}{-\omega^2 + \gamma j\omega + \omega_n^2}$, the inverse Laplace transform of which, with jω replaced by s, gives the stated impulse response.

Laplace transforms can be used to obtain a symbolic solution of the above initial value problem in terms of a convolution of its impulse response with a given forcing function.

The temporal solution of the initial value problem (13) is accordingly given by the convolution

$$x(t) = \frac{e^{-\gamma/2t} \sin(\omega_d t)}{\omega_d} * f(t) = \frac{\omega_n^2}{\omega_d} \int_0^t f(t-\tau) e^{-\frac{\gamma}{2}\tau} \sin(\omega_d \tau)\, d\tau . \qquad ...(15)$$

Since we are interested in resonators excited by periodic forcing function, we introduce the Fourier series of equ.(1), namely

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos(r\omega t) + b_r \sin(r\omega t) \right) \qquad ...(16)$$

into the above convolution integral, to obtain the result

$$x(t) = \frac{\omega_n^2}{\omega_d} \left[\frac{a_0 I_0(t)}{2} + \sum_{r=1}^{\infty} a_r I_r(t) + b_r J_r(t) \right], \qquad ...(17)$$

where

(i) $I_0(t) = \int_0^t e^{-\frac{\gamma}{2}\tau} \sin(\omega_d \tau)\, d\tau$,

(ii) $I_r(t) = \int_0^t e^{-\frac{\gamma}{2}\tau} \sin(\omega_d \tau) \cos(r\omega(t-\tau))\, d\tau$ and

(iii) $J_r(t) = \int_0^t e^{-\frac{\gamma}{2}\tau} \sin(\omega_d \tau) \sin(r\omega(t-\tau))\, d\tau$,

with r = 1,2,3, … .

The integrals (ii) and (iii) are respectively the real and imaginary parts of the integral $e^{jr\omega t}\int_0^t e^{-\left(\frac{\gamma}{2}+jr\omega\right)\tau}\sin(\omega_d\tau)\,d\tau$, where as before $j=\sqrt{-1}$ is the imaginary unit, whilst the integral in (i) is the value of (ii), when r = 0.

Putting $\frac{\gamma}{2}+jr\omega=\chi$, we have

$$\int_0^t e^{-\left(\frac{\gamma}{2}+jr\omega\right)\tau}\sin(\omega_d\tau)\,d\tau=\frac{\omega_d-\left(\omega_d\cos(\omega_d t)+\left(\frac{\gamma}{2}+jr\omega\right)\sin(\omega_d t)\right)e^{-\left(\frac{\gamma}{2}+jr\omega\right)t}}{\omega_d^2+\left(\frac{\gamma}{2}+jr\omega\right)^2}, \qquad \text{...(18)}$$

so that

$$\begin{aligned} I_r+jJ_r&=\int_0^t e^{-\frac{\gamma}{2}\tau}\sin(\omega_d\tau)e^{jr\omega(t-\tau)}=\frac{R+Xj}{\omega_n^2-r^2\omega^2+j\gamma r\omega},\\ &=\frac{(\omega_n^2-r^2\omega^2)R+\gamma r\omega X+j(\omega_n^2-r^2\omega^2)X-\gamma r\omega R}{(\omega_n^2-r^2\omega^2)^2+\gamma^2r^2\omega^2}, \end{aligned}$$

...(19)

where $R=\omega_d\cos(r\omega t)-\left(\omega_d\cos(\omega_d t)+\frac{\gamma}{2}\sin(\omega_d t)\right)e^{-\frac{\gamma}{2}t}$ and $X=-r\omega e^{-\frac{\gamma}{2}t}\sin(\omega_d t)$.

Equating real and imaginary parts in the equ.(14) then gives

$$I_r=\frac{(\omega_n^2-r^2\omega^2)\omega_d\cos(r\omega t)-\left(\omega_d\cos(\omega_d t)+\gamma\left(\frac{1}{2}+r^2\omega^2\right)\sin(\omega_d t)\right)e^{-\frac{\gamma}{2}t}}{(\omega_n^2-r^2\omega^2)^2+\gamma^2r^2\omega^2} \qquad \text{...(20)}$$

and

$$J_r=\frac{r\omega\left[-\frac{\gamma^2}{2}\omega_d\cos(r\omega t)+\left[\gamma\omega_d\cos(\omega_d t)-\left(\omega_n^2-r^2\omega^2-\frac{\gamma^2}{2}\right)\sin(\omega_d t)\right]e^{-\frac{\gamma}{2}t}\right]}{(\omega_n^2-r^2\omega^2)^2+\gamma^2r^2\omega^2}. \qquad \text{...(21)}$$

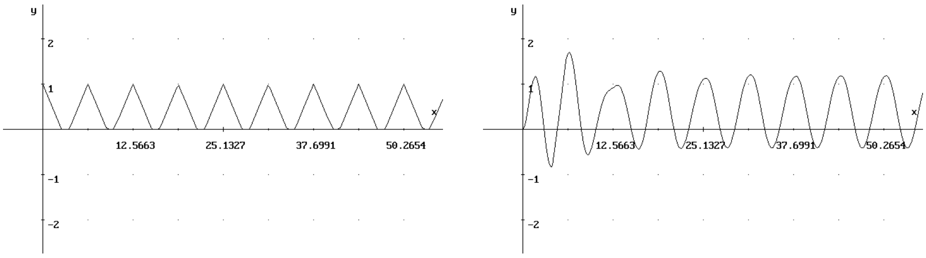
Finally r = 0 in equ.(18) above, yields

$$I_0=\frac{\omega_d}{\omega_d^2+\left(\frac{\gamma}{2}\right)^2}-\frac{\left(\omega_d\cos(\omega_d t)+\frac{\gamma}{2}\sin(\omega_d t)\right)e^{-\frac{\gamma}{2}t}}{\omega_d^2+\left(\frac{\gamma}{2}\right)^2}. \qquad \text{...(22)}$$

Fig. 15 shows graphs of the input and output based on the above theory, for the case in which the input is the glottal triangular wave, described by the Fourier series in equ.(12) with coefficients

$a_0 = \dfrac{\alpha}{\pi}$ and $a_r = \dfrac{2(1-\cos(r\alpha))}{\pi\alpha r^2}$, where for test purposes only, the parameters in the input graph are given by $\alpha = 5\pi/6$, and $T = 2\pi$.

The output corresponds to that of a damped oscillator with damping factor $\gamma = 0.25$ and a natural angular frequency of test value $\omega_n = 3/2$ radians per second.



Input Output

Fig. 15

From the graph we see that apart from a slight glitch at the beginning of the motion, it ultimately settles down to a steady oscillation.

An alternative treatment of formant resonance may be conducted by comparing the aerial dynamics with that of the longitudinal displacement of air in pipes. If the cross-sectional radius of a given pipe is small compared with the wavelength, it transpires that the sound waves may be treated as plane waves, for which particle displacements $\xi(x, t)$ again satisfy the wave equation (7), where c is now the speed of sound as described in the sequel to equ.(14).

The general solution for a pipe of length ℓ, depends on the boundary conditions.

For a pipe closed at both ends, the displacement $\xi(x,t)$ at station x and time t is given by

(i) $$\xi(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi ct}{\ell}\right).$$

However this is not a practical proposition since in singing the mouth needs to open, except in the case of humming when the nostrils constitute the open end of a more complex structure.*

Again if the pipe is open at both ends the general solution is

(ii) $$\xi(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi ct}{\ell}\right).$$

Actually the best configuration would be for a pipe open at one end - the mouth - and closed at the other end - the throat, when the general solution is

*Apart from nasalised vowels, the nasal cavities are excluded when singers raise the velum, as in a simulated yawn, whilst vocalising.

(iii) $$\xi(x,t)=\sum_{n=0}^{\infty} C_n \sin\left(\frac{2n+1}{2}\cdot\frac{\pi x}{\ell}\right)\sin\left(\frac{2n+1}{2}\cdot\frac{\pi ct}{\ell}\right).$$

The temporal sine functions occurring in the above standing waves are a consequence of assuming the motion starts from rest, in which open end corrections have been ignored. The pitches of the gravest modes in solutions (ii) and (iii) above are respectively given by $c/2\ell$ and $c/4\ell$, which indicates that a pipe stopped at one end 'blows' an octave lower than a completely open pipe.

For a formant model based on the solution (iii), the successive terms in the expansion for ξ constitute the resonances of the acoustic tube.

For a tube of length ℓ = 17 cm, which is representative of a male-voice vocal tract [22], the first four formant resonances of a single but fixed hypothetical vowel type sound, may be tabulated as in table 1 below.

Formant	n	ℓ	ν		ℓ = 17 cm
F_1	0	$\lambda_0/4$	$c/4\ell$	=	500 Hertz
F_2	1	$3/4\ \lambda_1$	$3c/4\ell$	=	1500 Hertz
F_3	2	$5/4\ \lambda_2$	$5x/4\ell$	=	2500 Hertz
F_4	3	$7/4\ \lambda_3$	$7c/4\ell$	=	3500 Hertz

Table 1

The fact that pressure is related to particle displacements by $p=-\rho c^2\frac{\partial\xi}{\partial x}$ enables us to illustrate the above data by means of the corresponding pressure profiles in Fig. 16 on the following page.

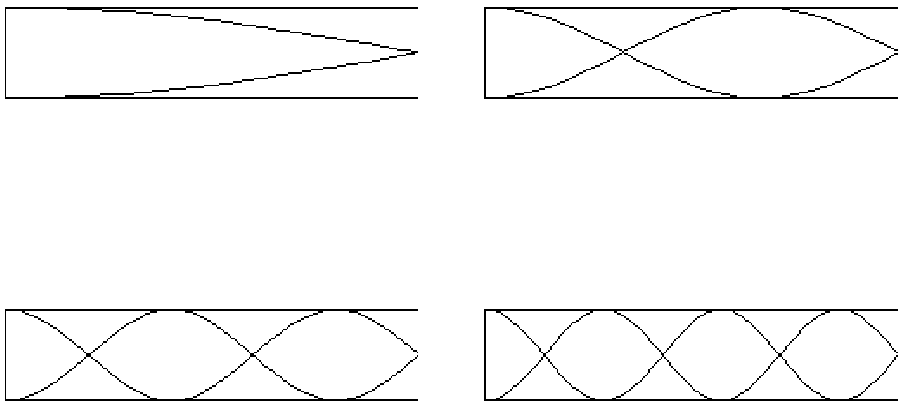


Fig. 16

It is nice that such an elementary model of the vocal tract as the acoustic tube capably predicts formant frequencies of the right sorts of magnitude.

Perhaps a model based on Webster's horn equations incorporating a variable cross-section for the vocal cavities might be effective in allowing for variable vowel sounds [26].

In such a case the displacement variable ξ needs to be replaced by the variable X/A, where X represents the volume and A the cross-sectional area of the vocal tract at station x and time t. Introducing the 'variables separable' expressions for pressure p and X, namely $p = \hat{p}(x)e^{j\omega t}$ and $X = \hat{X}(x)e^{j\omega t}$, into Webster's equations, leads to the spatially dependent equations: (i) $\frac{1}{A}\frac{d}{dx}\left(A\frac{d\hat{p}}{dx}\right)+k^2\hat{p}=0$, and (ii) $A\frac{d}{dx}\left(\frac{1}{A}\frac{d\hat{X}}{dx}\right)+k^2\hat{X}=0$, with $\hat{p}=-\frac{\rho c^2}{A}\frac{d\hat{X}}{dx}$.

These equations are furthermore subject to the compatibility condition $A(x)=A_0\cosh^2\left(\frac{x}{h}+\beta\right)\mathrm{sech}^2(\beta)$, where β, h and A_0 are constants [18,19].

The third and fourth formants are weak compared with the first two, so vowels can be effectively distinguished by their first and second formants. These are called the base and hub respectively. Vowels with widely separated base and hub may be classed as bright and are called front vowels since the tongue-hump is located towards the front of the mouth. Vowels with moderate separation of base and hub are produced when the tongue-hump is towards the middle of the mouth and are called central vowels for this reason. Placing the tongue-hump towards the back of the mouth produces the dark vowels with base and hub closer in pitch.

A guide to the characteristics of vowels may be obtained from the scheme below. The musical entries correspond to the average frequencies of the base and hub of a given vowel. Below each musical entry is a picture of the sort of output one would expect from a speech spectrograph, together with a diagram depicting the position of the tongue in the production of the vowel.

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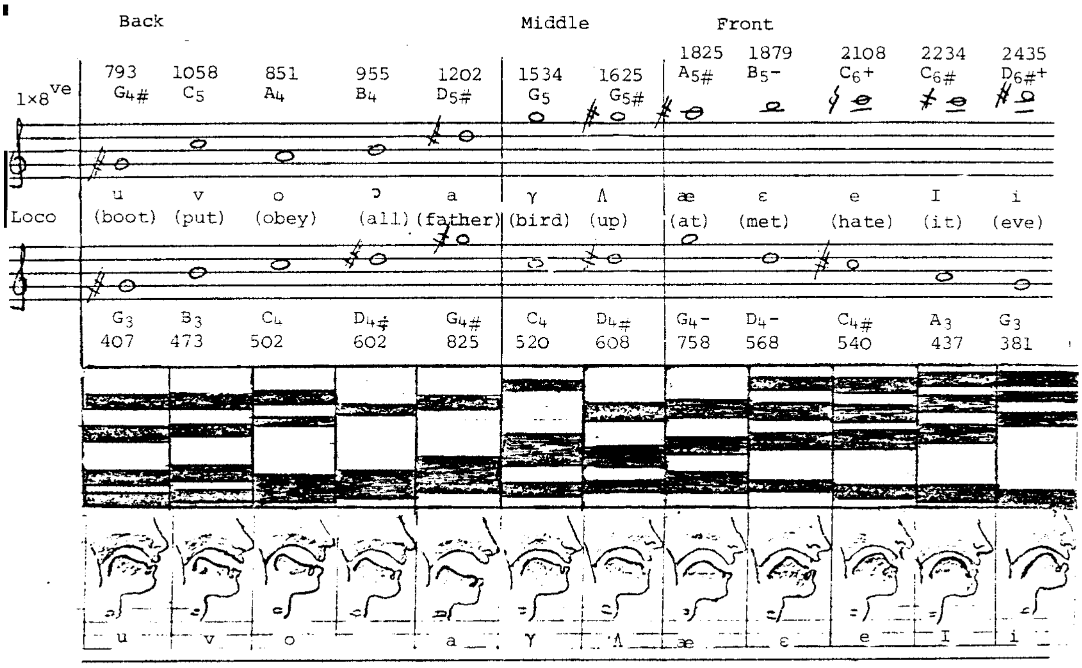

High back | Low back | Low front | High front

Fig. 17

It is customary to plot the hub against the base frequency-wise, for different vowels, as indicated in Fig. 18. This plot can be used to indicate the locus of tongue-hump movements in articulating the various vowels, as depicted next in Fig. 19 [23].

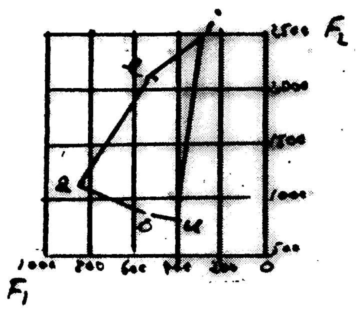

Fig. 18

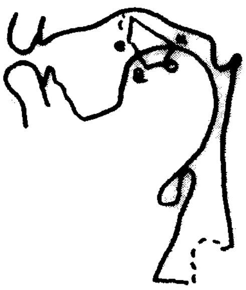

Fig. 19

Diagrams such as the above have applications in speech therapy for the deaf [24,25].

It is instructive to conclude this section by pointing out that adiabatic formula $c = \sqrt{\gamma p_0 / \rho_0}$, for the speed of sound in air, predicts that warm wind instruments will play sharp, since the density of air in such cases will be less than that for a cold instrument. This means that the speed of sound will increase with the warmth of the instrument. At the same time c = νλ, where λ is the wavelength (which stays constant with the length of the instrument)* and ν is the frequency of the sound, it follows that the frequency must increase with the warmth of the

*The coefficient of linear expansion having negligible effect in this case.

instrument. This effect can in fact be quite significant, as for example for a wind band playing on a hot summer's day inside an even hotter marquee tent; when the instruments require careful retuning by mechanically increasing the length of each instrument according to design.

In fact even initially cold instruments warmed by the player's breath become noticeably sharper.

Sound and Transmission

When an object vibrates, it induces the molecules of the surrounding air to oscillate back and forth in sympathy. If ε denotes the molecular displacements from their equilibrium position and t is time, the equation of motion of their macroscopic continuum, on the adiabatic hypothesis, is a non-linear wave equation of the form

$$\frac{\partial^2 \varepsilon}{\partial t^2} = c^2 \frac{\partial^2 \varepsilon}{\partial x^2}\left(1 + \frac{\partial \varepsilon}{\partial x}\right)^{\gamma+1}, \qquad \text{...(23)}$$

where c is the speed of sound in air, as previously described. A second order approximation to the above non-linear wave equation in equ.(23) above is

$$\frac{\partial^2 \varepsilon}{\partial t^2} = c^2 \frac{\partial^2 \varepsilon}{\partial x^2} - (\gamma+1)c^2 \frac{\partial \varepsilon}{\partial x}\frac{\partial^2 \varepsilon}{\partial x^2} \qquad \text{...(24)}$$

for which a particular solution is

$$\varepsilon = f\left(t - x/c\right) + \frac{\gamma+1}{4c^2} x \left(f\mathrm{S}\left(t - x/c\right)\right)^2, \qquad \text{...(25)}$$

where ε = f(t) is the wave profile at x = 0, [4].

Substituting the simple combination

$$f(t) = a_1 \cos\left(\omega_1 t\right) + a_2 \cos\left(\omega_2 t\right) \qquad \text{...(26)}$$

into equ.(23) demonstrates the existence of frequencies equal to the difference, sum and octaves of ω_1 and ω_2, with amplitudes proportional to the squares and products of the original amplitudes a_1 and a_2, showing that these derived tones increase in relative importance to the parent tones.

This supports the idea that additional tones are created in the transmission of harmoneous sounds.

Sound detection

When sound waves enter a listener's ears, they impinge on the ear drums which are made to vibrate in sympathy. The restoring force on the ear drum in being displaced a distance x from its equilibrium position by a given wave, takes the form

$$F = \omega^2 x + \alpha x^2, \qquad \text{...(27)}$$

where ω is a representative angular frequency of the aerial waves comprising the vibrations and α is a small constant. The vibratory motion of the ear drum is thus described by an anharmonic motion equation of the form

$$\frac{d^2x}{dt^2}+\omega^2 x=-\alpha x^2, \qquad ...(28)$$

where x may be regarded as the displacement of the drum from equilibrium and t is again time. Assuming the initial zero order approximation

$$x = x_0\cos(\omega t) \qquad ...(29)$$

in the face of the initial conditions $x(0) = x_0$ and $\dot{x}(0)=0$, it can be shown via perturbation methods [6], that a solution to equ.(28), correct to the second order in the small quantity α, is

$$\begin{aligned} x=&-\frac{\alpha {x_0}^2}{\omega_0^2}\left(\frac{1}{2}+\frac{\alpha x_0}{3\omega_0^2}\right)+x_0\left(1+\frac{\alpha x_0}{3\omega_0^2}+\frac{29}{144}\frac{\alpha^2 x_0^2}{\omega_0^4}\right)\cos(\omega_0 t) \\ &+\frac{\alpha x_0^2}{3\omega_0^2}\left(\frac{1}{2}+\frac{\alpha x_0}{3\omega_0^2}\right)\cos(2\omega_0 t)+\frac{\alpha^2 x_0^3}{48\omega_0^4}\cos(3\omega_0 t), \end{aligned} \qquad ...(30)$$

where $\omega^2=\omega_0^2+\dfrac{5}{6}\dfrac{\alpha^2 x_0^2}{\omega_0^2}$ [5-11] ...(31)

In this we see that the amplitude of each harmonic component depends on the frequency of that component, which is typical of non-linear systems.

An analytical solution to the above initial value problem in (27) and (28) can be found in terms of Jacobian elliptic and hyperbolic functions. In pursuit of this solution it is expedient to first non-dimensionalise equs. (27) and (28) by replacing the displacement x by x/x_0 and time t by ωt, when the equations become $\dfrac{d^2x}{dt^2}+x=-\dfrac{3\varepsilon}{2}x^2$, and x(0) = 1, where $\varepsilon=\dfrac{2\alpha x_0}{3\omega^2}$. The analytic solution is then [13,14]

$$x=1-Q^2 sn^2\left(P\sqrt{\varepsilon}\,t/2,\,Q/P\right),^* \qquad ...(32)$$

where (i) $P^2=1-x_2$, (ii) $Q^2=1-x_1$,

with $x_2 < x_1$ being the zeros - always exceeded by unity - of the quadratic expression

$$x^2+\left(1+\frac{1}{\varepsilon}\right)x+\left(1+\frac{1}{\varepsilon}\right).$$

---

*or equivalently $x=1-PQ\left(\dfrac{1-\mathrm{cu}\left(\sqrt{PQ\varepsilon}\ t,\dfrac{P+Q}{2\sqrt{PQ}}\right)}{1+\mathrm{cu}\left(\sqrt{PQ\varepsilon}\ t,\dfrac{P+Q}{2\sqrt{PQ}}\right)}\right)$, [15].

Plots, for various values of ε are shown in Fig. 20.

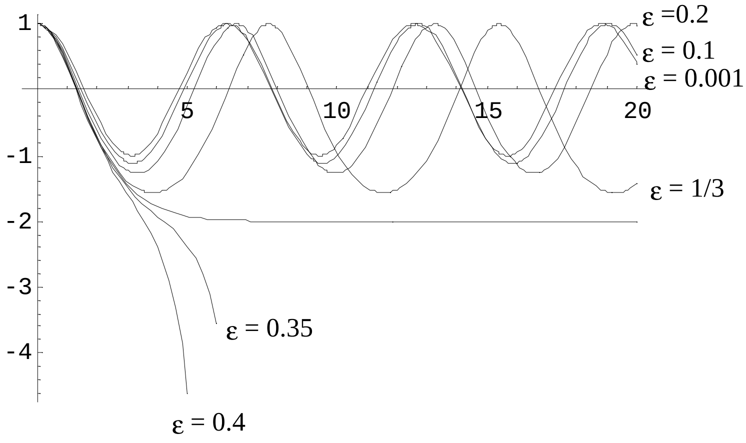

Fig. 20

These plots indicate that only when $\varepsilon < 1/3$ is the solution oscillatory, with a fundamental angular frequency given by

$$\omega_0 = P\sqrt{\varepsilon}/F\,, \qquad ...(33)$$

where

$$F = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k_1^2\right) = 1 + k^2/4 + 9k_1^4/64 + 25k_1^6/256 + \cdots \qquad ...(34)$$

is a hypergeometric function.

When $\varepsilon = 1/3$ the solution reduces to $x = 1 - 3\tanh^2(t/2)$, whilst for $\varepsilon > 1/3$ the solution becomes unbounded.

The periodic solution for ε < 1/3 in equ. (32), can accordingly be expanded into a Fourier series, which, on returning to the original dimensional variables, takes the form

$$x = x_0 - \frac{Q^2(F - F_1)}{k^2 F} + \frac{8Q^2}{k^2F^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos(n\omega_0 t)\,, \qquad ...(35)$$

where $q = e^{-\pi K'/K}$, in terms of the complete elliptic integrals $K = \frac{\pi}{2}F$ and $K' = \frac{\pi}{2}F_1$ of the first kind, where

$$F_1 = {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;k_1^2\right) = 1 - k^2/4 - 3k^4/64 - 5k^6/256 + \cdots\,. \qquad ...(36)$$

If the various parameters in the above Fourier series are expanded as infinite series in ε, the equations (29) and (30) emerge as second order approximations to the solution for ε < 1/3.*

*See also [16,17] for more on the anharmonic motion equation.

The above analysis suggests that the tympanic membrane invents additional virtual harmonics of slightly lower frequencies in its oscillatory response to simple tones. These vibrations are subsequently transmitted via the oscicles of the inner ear to the cochlea, where oscillatory motions are induced in a fluid, the various frequencies of which are detected by sensitive hair cells, which provide a mechanism for converting the vibrations to nerve impulses, to be ultimately interpreted by the listener's brain as sound.

It is interesting to speculate that the solution for $\varepsilon > 1/3$, being unbounded, indicates that the anharmonic motion model predicts the possibility of damage to the ear drum for ε in this range.

Normal hearing requires the lowest frequency of vibration to be at least 20 Hz, whilst the highest perceptible frequency is of the order of 20 kHz. Actually, the ear senses the intensity (power per unit area) of a sound wave, rather than just the amplitude, where the sound intensity of a given wave is proportional to the square of the product of its amplitudes and frequency.

Scales

Melody derives from the fact that the human ear perceives a doubling of an oscillation's frequency as an octave. Hence by a simple fractioning process based on halving or doubling the various individual frequencies present in a Fourier series, with no regard to the amplitude, melodic sequences of notes can be produced which can be given greater variety by varying the duration of each note within the sequence.

The orchestral tune up note of concert A, indicated in Fig. 21, is defined to have a secular frequency of 440 Hz.

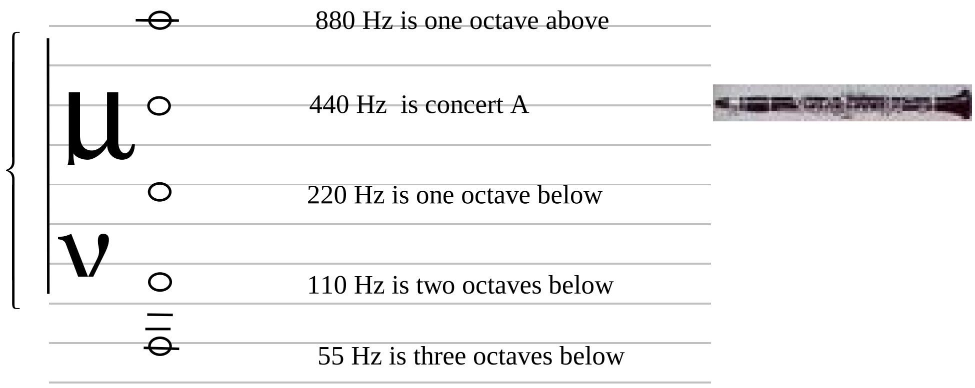

Fig. 21

Doubling the frequency of a note puts the result one octave (Ω) above. Thus 888 Hz, as shown in Fig. 21, is one Ω above concert A.

Likewise halving the frequency gives a note one Ω below, whilst quartering the frequency makes a note 2Ω below etc., as also indicated in Fig. 21.

Since the secular frequency is the reciprocal of the period T, we see that the period of concert A can be obtained by setting $1/T = 440$ Hz, whence $T = 1/440$ sec.

Furthermore the angular frequency of concert A would be $\omega = 2\pi \times 440 = 880\pi$ rads per sec; not to be confused with 880 Hz, which is the secular frequency of the note one octave above concert A.

The corresponding simple vibration of unit amplitude for the pure concert A tone would then be given by $y = \sin(880\pi t)$.

This is a very fast oscillation to plot graphically and if we are not careful in our choice of the time interval for the plot, we just get a black smudge as indicated in the left hand graph of Fig. 22, where the duration of the oscillation - call it a time window - is π seconds.

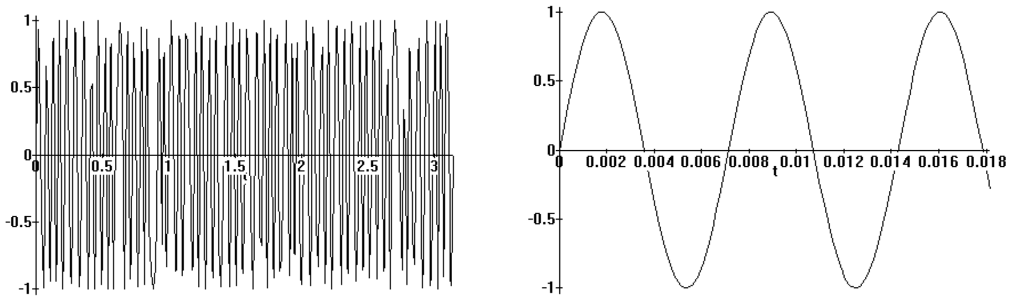

Fig. 22

To get a better picture we must reduce the time window to a duration of say, 1/55 sec, when we get the recognisable sine curve in the right hand frame of Fig. 22 above.

Let us now indicate more of the successive harmonics obtained by doubling, tripling and quadrupling the fundamental frequency of 66 Hz as in Fig. 23, together with the various other attributes and graphs etc., associated with these harmonics. *

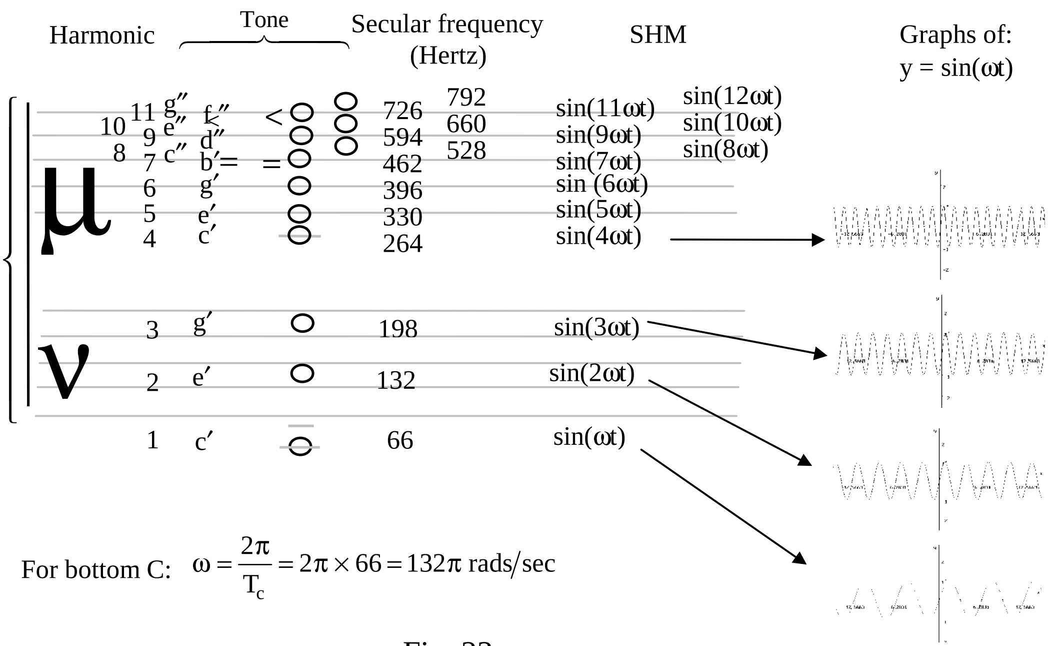

Fig. 23

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*Jazz musicians familiar with guitar chord symbolism will recognise that the spread of harmonics up to the eleventh, in the key of C, is nothing more than $C_{9\ sus\ 11+}$.

By continually halving the frequency of an appropriate tone in Fig. 23, we can place the resulting tone in the same register as that of the fundamental frequency, so producing a diatonic scale. Four of the six notes in the diatonic scale, i.e. those excluding the keynote and its octave may be obtained in this way.

Thus the supertonic being three octaves above the 9th partial bears a frequency ratio to the fundamental of $9/2^3 = 9/8$. The corresponding ratios for the other notes including the mediant, dominant and leading note work out similarly to 5/4, 3.2 and 15/8 respectively. This produces the justly tuned scale indicated in Fig. 24.

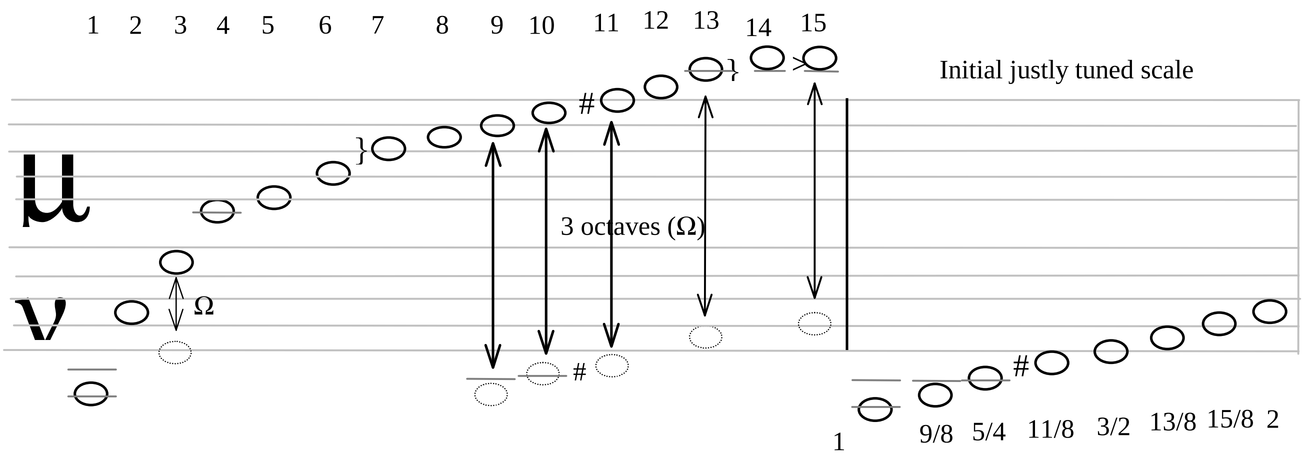

Fig. 24

In contrast the diatonic scale is produced if the subdominant and submediant of the scale are obtained by doubling the frequencies of the notes one fifth and a minor third below the keynote respectively. Accordingly the frequency ratio corresponding to a “fifth above” is 3/2, so the ratio for fifth below is the reciprocal of this, namely 2/3. Doubling this gives 4/3 as the frequency ratio of the subdominant. In the same way the frequency ratio of the submediant is calculated as 5/3.

The justly tuned scale so refined constitutes the so-called diatonic scale, which is indicated in Fig. 25.

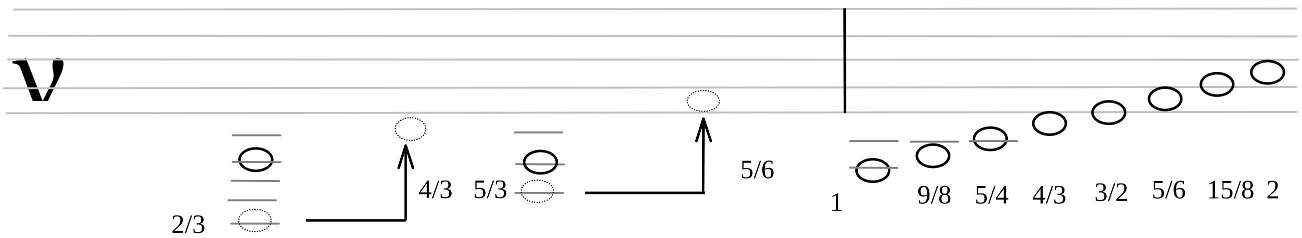

Fig. 25

It is interesting to compare the tuning discrepancies that exist between the various refined notes and their original counterparts as in the frequency ratios (i) $\dfrac{\textit{just submediant}}{\textit{diatonic submediant}} = \dfrac{39}{40}$ and (ii) $\dfrac{\textit{just subdom.}}{\textit{diatonic subdom.}} = \dfrac{33}{32}$. Further tuning discrepancies are considered in [20].

Although the ratios of the frequencies of consecutive notes in the diatonic scale are of the form $(n+1)/n$ for various whole number n (i.e. the ratio from e to f would be $\frac{4/3}{5/4}=\frac{16}{15}$, musicians tend to talk of whole or half steps (dubbed tones and semi-tones respectively), there being five such whole steps and two such half steps within the octave. Again the number of steps between pairs of successive notes, called intervals, are designated ordinally. Thus the intervals between c and e, comprising three whole steps, is called a major third. Subtracting a semitone from this results in a so-called minor third.* Likewise the interval between C and F comprising three and a half steps is called a fourth. Adding a semi-tone results in a so-called augmented fourth, which coincides approximately with the diminished fifth, which is the interval of a fifth (C to G say) minus one semi-tone.

A notable fundamental refinement which introduces a scale of equal half step intervals, leads to the alternative scheme in which the octave between 110 and 220 Hz, is split into twelve semi-tone steps. This can be done by multiplying the lower tone 110 Hz, by the successive integral powers of the twelfth root of two. Better still, multiplying 110 Hz in turn by each one of the members of the sequence $\{r_n\}=\{2^0, 2^{1/12}, 2^{2/12}, 2^{3/12}, \cdots, 2^{23/12}, 2^{24/12}\}$, produces 24 equally tempered semi-tones covering the two octaves from 110 to 440 Hz, as depicted in Fig. 26, where the numbers inside the semibreves indicate the power to which $2^{1/12}$ has been raised to produce the appropriate frequency ratio between the circled note and the fundamental tone of 110 Hz.

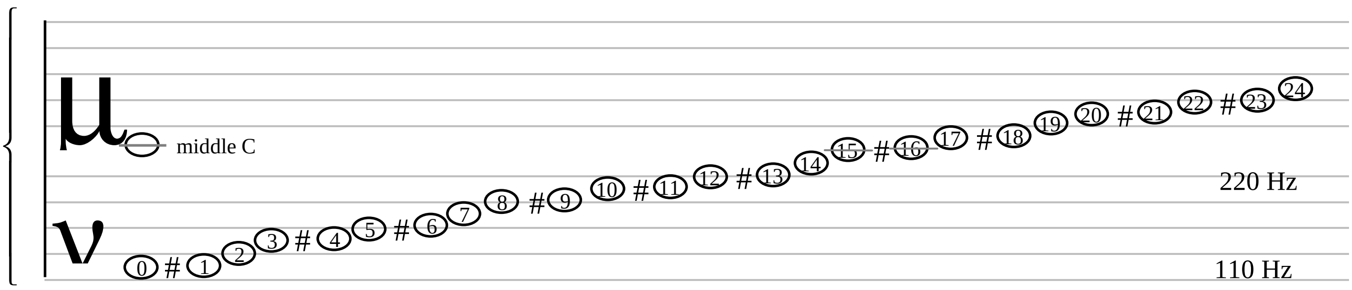

Fig. 26

The corresponding sequence of secular frequencies is then $\{v_n\}=\{110r_n\}$, whilst the sequence of angular frequencies is in turn, given by $\{\omega_n\}=\{2\pi v_n\}$.

The sequence of musical notes now corresponds to the scheme $\{tone(n)\}=\{\sin(\omega_n t)\}$. Not only can these sinusoids in the sequence be plotted by a symbolic package, but they can also be played by packages like **MATHEMATICA**.

It is useful at this juncture to note that in the above equi-tempered scheme, the various intervals previously alluded to, now encompass appropriate numbers of semi-tone steps, involving no tuning discrepancies arising from the adopted musical path.

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*In the stride piano styles of Teddy Wilson, Fats Waller etc., the left hand often combines the ‘vamping’ of chords with a chromatic movement in parallel tenths.

Syncopation

The duration of notes in a melody may be controlled by means of Heaviside's unit step function which is defined by

$$H(t) = \begin{cases} 0 : t < 0 \\ 1 : t > 0 \end{cases}, \qquad \text{...(36)}$$

and is graphed in Fig. 27.*

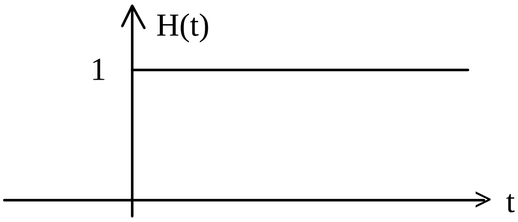


Fig. 27

The Heaviside step function is an idealisation of a switching function or an 'if' statement, which changes instantaneously but discontinuously, at time t = 0, from 'off' (zero) to 'on' (one).

A time delay of amount a may be introduced into the step function, by writing

$$H(t-a) = \begin{cases} 0 : t-a < 0 \\ 1 : t-a > 0 \end{cases} = \begin{cases} 0 : t < a \\ 1 : t > a \end{cases}. \qquad \text{...(37)}$$

This delayed step has the graph depicted in Fig. 28.

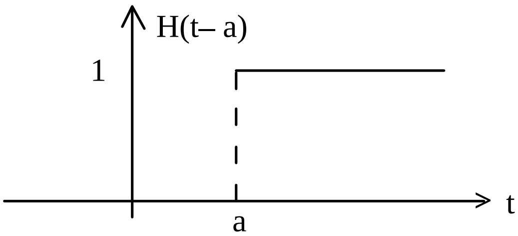


Fig. 28

Subtracting two delayed step functions, gives a pulse, as in

$$H(t-a) - H(t-b) = \begin{cases} 0 : t < a \\ 1 : a < t < b \\ 0 : b < t \end{cases}; \qquad \text{...(38)}$$

with the graph indicated in Fig. 29.

---

*A more sophisticated definition is: $\mathrm{H(t)} = \begin{cases} 0 & : \mathrm{t} < 0 \\ 1/2 & : \mathrm{t} = 0, \\ 1 & : \mathrm{t} > 0 \end{cases}$ but we will have no occasion to use this.

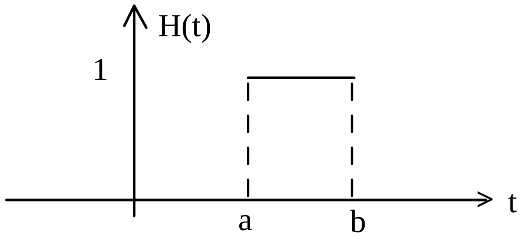


Fig. 29

We can finally set up a sequence of durations by defining, in terms of delayed Heaviside step functions, the pulse functions $time(m,n) = H(t - mp) - H(t - np)$, where for a given value of p, the duration of the pulse will be (n – m)p seconds, assuming n > m ≥ 0.

Melody

The fact that the difference between two consecutively delayed step functions constitutes a pulse can be used in conjunction with periodic functions like sinusoids for instance, to construct the mathematical analogue of a melody.

Let us apply the above idea to simulate mathematically, the first three notes of the nursery rhyme 'Three Blind Mice', say in the key of C.

Rather than invoke the frequencies of Fig. 15 as they stand - implicitly in the key of A major, we will transpose the entire scheme up to the key of C, by multiplying our original sequence of frequencies by $2^{3/12} = 2^{1/4}$, as indicated in Fig. 30.

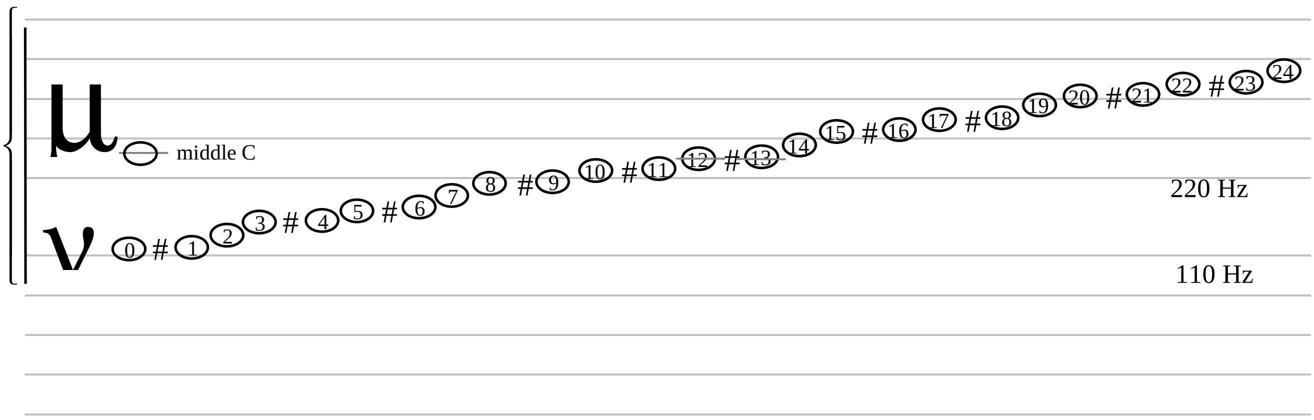
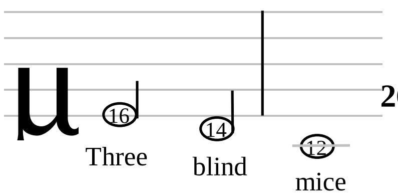


Fig. 30

The required notes of three blind mice as extracted



from the scheme in Fig. 30, are shown in Fig. 31.

Defining in turn the elements $\nu_n = 2^{1/4} \times 110 r_n$, $\omega_n = 2\pi\nu_n$ and $tone(n) = \sin(\omega_n t)$. we can construct the vector of tones
<u>tones</u>(t) = (tone(16), tone(14), tone(12)).

Letting p = 1/2, corresponding to half a second, we can likewise construct a vector of note durations, thus <u>duration</u> = (time(0, 1), time(1, 2), time(2, 3)). Forming the scalar or dot-product of these two vectors gives the tune or lead, as lead = <u>tone</u> · <u>duration</u> , which is plotted as a temporal function in Fig. 32 next.

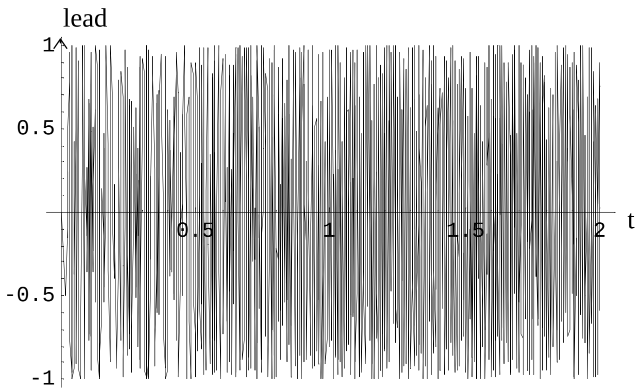


Fig. 32

Although this plot is not very edifying, it is included on the basis of 'what we can plot we can play' (WWCPWCP).

Harmony

In a similar manner we can construct a tenor part to harmonise with the lead above it, as in Fig. 33. This part is summarised by the vector <u>notes</u> = (tone(19), tone(17), tone(16)) with the duration vector the same as before, so that the tenor part is again tenor = <u>notes</u> · <u>duration</u>. This is plotted in Fig. 34.

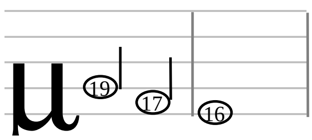


Fig. 33

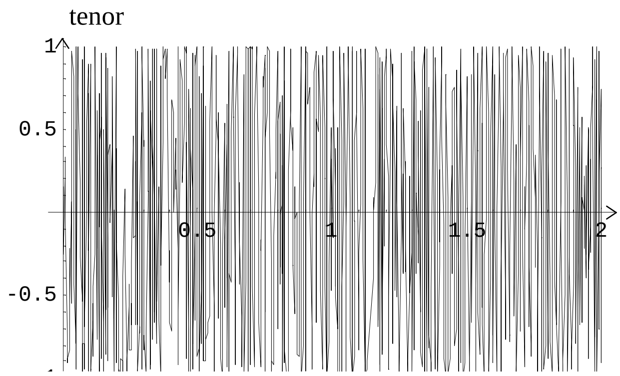


Fig. 34

A linear combination of these parts which accentuates the lead at the expense of the tenor, namely tenor + lead = 1.5 lead + 0.9 tenor, constitutes the two part harmony shown in Fig. 35, with the corresponding plot indicated in the accompanying graph of Fig. 36.

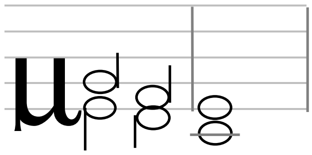


Fig. 35

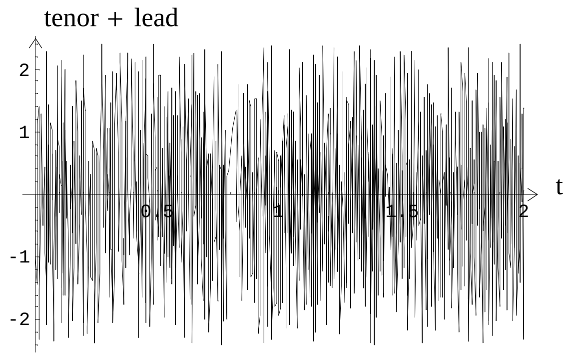


Fig. 36

We can likewise construct the baritone part indicated in Fig. 37. Describing this by the vector <u>notes</u> = (tone(12), tone(11), tone(7)) and 'dotting' with the duration vector gives the function of t plotted in the accompanying Fig. 38.

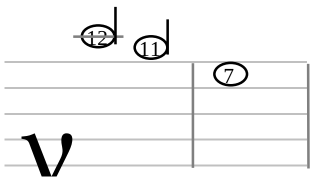


Fig. 37

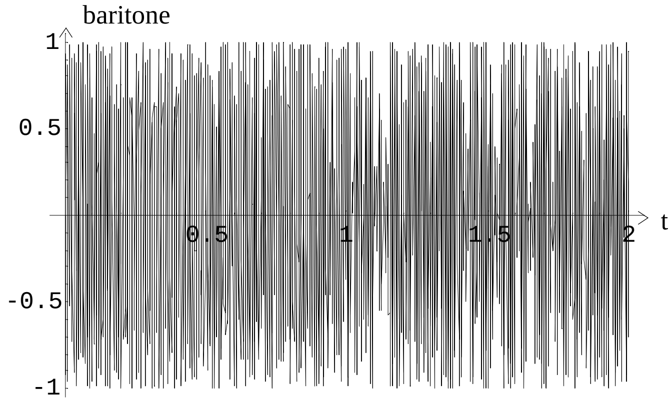
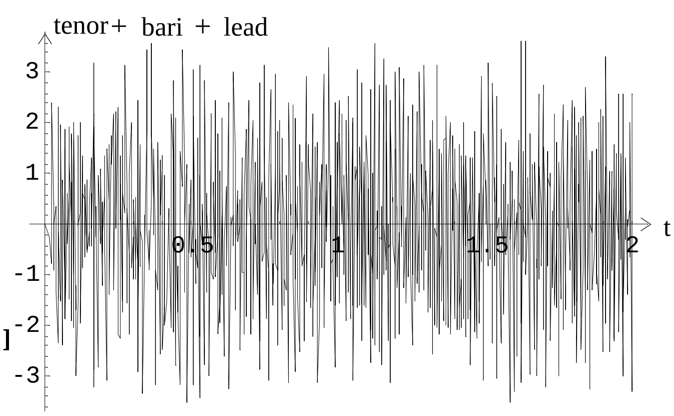


Fig. 38

The three parts can again be combined according to say

tenor + bari + lead = 1.5lead + 0.9tenor + 1.3bari, to give the three part harmony indicated in Fig. 39 and graphed in Fig. 40.



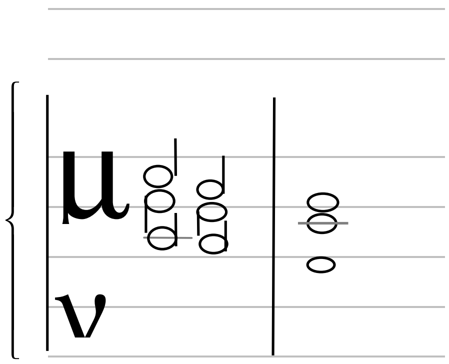


Fig. 39

We can finally in the same manner as before, add the base part indicated in Fig. 41, with its accompanying plot in Fig. 42 to get the four part (barbershop voiced) harmony with its graph, shown in Figs. 43 and 44 respectively.

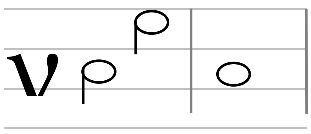


Fig. 41

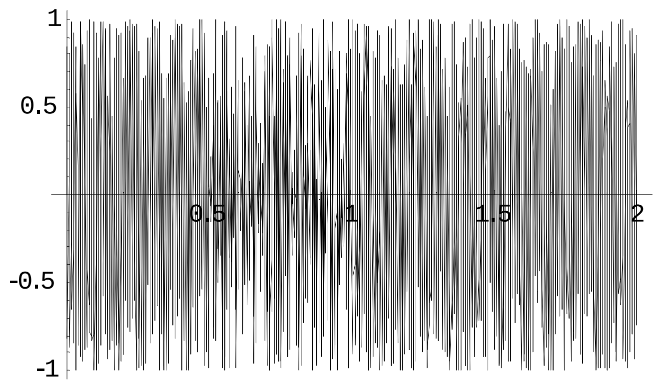


Fig. 42

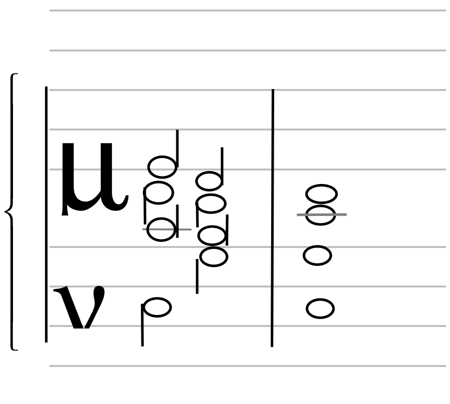


Fig. 43

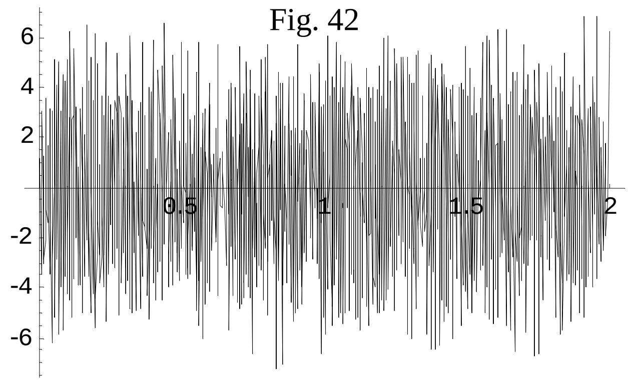


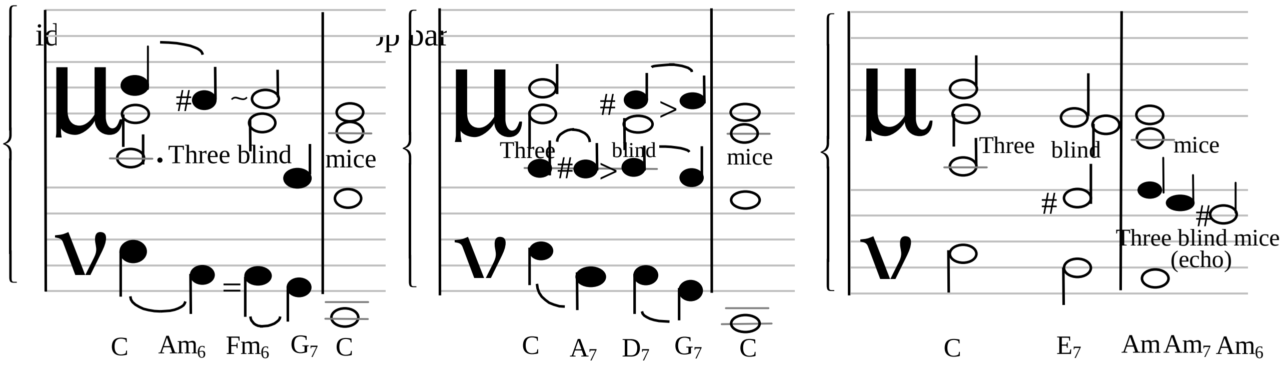
Fig. 44

In **MATHEMATICA** the above plots can be converted to sound by simply replacing the **plot** command with the **play** command, so that we can in fact listen to the above melodies and harmonies.

Harmonic highways

Seasoned barbershoppers would find the above harmonies rather mundane, so they would perhaps look for more interesting harmonic highways, containing perhaps sequences of chromatic modulations based on the so-called cycle or circle of fifths, this being predominantly a succession of dominant seventh chords following a predetermined path to resolution.

Possible harmonic sequences with contrapuntal overtones for the 'Three Blind Mice' clip, are indicated in Fig. 45, with an additional little embellishment in the last graph, illustrating the



Via the supertonic ninth Via the submediant seventh Via the medient seventh

Fig. 45

It is left as an exercise for the reader to construct the appropriate scalar products of the above notes and duration vectors to produce the time based functions and their graphs which can then be played in **MATHEMATICA**.

Concluding Remarks

The tag indicated in Fig. 43 was programmed into **MATHEMATICA** with the sine waveforms replaced by the output waveform in the right hand frame of Fig. 15, containing different values of γ and ω_n which were chosen from Fig. 17 and assigned for each vertical chord, with a view to mimicking the first formant only of certain selected vowel sounds.

The computational effort was immense!

At the time **MATHEMATICA** version three was being used without success. The entire coding was subsequently E-mailed to Support@wolfram.com, where it took a Windows machine with a speed of 300 M and 64 MB of RAM somewhere around a couple of days for **MATHEMATICA** version 4 to render the functions being worked upon. However once the

[*]Barbershoppers actually sing short snippets like the above. They call them barbershop tags.

wavefile was created its existence ensured only a matter of seconds in which to hear the noise.

Unfortunately the sound is somewhat marred by the odd click or pop. There may also be aliasing errors, poor quality speakers etc. distorting it, otherwise the effect is quite pleasing.

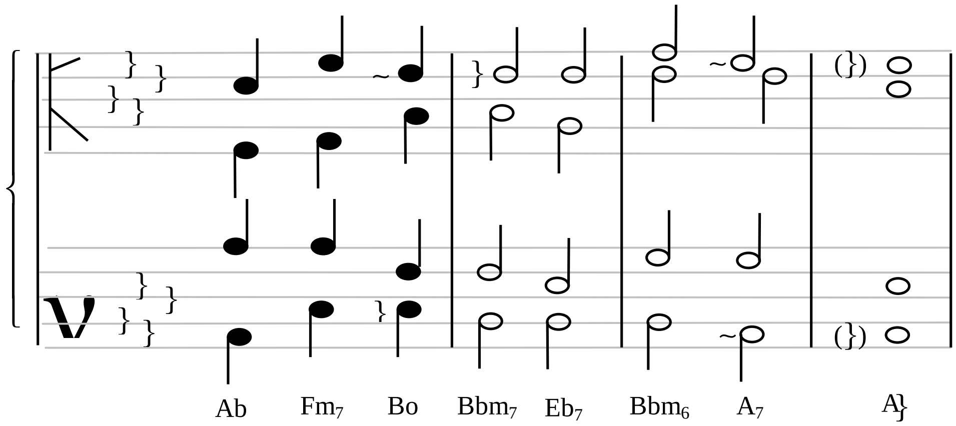


Fig. 46

Future work

The idea is to repeat the above exercise with four damped harmonic motion equations coupled together as a system of ordinary differential equations, in an attempt to capture the four formants of various vowel sounds.

It would also be interesting to replace the equi-tempered notes by their justly tuned diatonic counterparts to optimise the synergistic effects of having whole-number frequency ratios in the harmonics.

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