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Towards a Theory of Practices For Teaching and Learning Mathematics with CAS

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Introduction

If we want to capture the changes that technology has brought into mathematics classrooms, we need to have a broad perspective regarding the processes of teaching and learning mathematics. The recent work of Chevallard (1992, 1997) proposes a common framework that can be used to make comparisons between the traditional and emerging practices of mathematics teaching and learning. Chevallard assumes that mathematics education as a domain of research is part of an 'anthropology of knowledge', which studies the emergence, growth, and functioning of what is known as 'knowledge' in various cultures. Didactics of mathematics is the part concerned with mathematical knowledge and the processes of its emergence, growth, and functioning in teaching mathematical topics. A topic can be characterized by its practices, or rather by its theory of practices, i.e. a *praxeology*. A *praxeology* of the teaching of a given topic can be described, according to Chevallard, by a consistent set of four components:

Tasks that it is supposed to accomplish;
Teaching techniques – know-how for accomplishing the tasks;
Technology – a discourse justifying the teaching techniques;
Theory – a proposed conceptual basis for the discourse.

This framework of *praxeology* was applied in analyzing the formative development research of the MathComp project. The MathComp (mathematics on computers) project began in 1996 in the Science Teaching Department at the Weizmann Institute of Science. Its aim is to integrate Computer Algebra Systems (CAS) into teaching, to improve the learning of mathematics by creating a network of relations between mathematical ideas, concepts, and the procedures leading to them. CAS does not answer questions; rather, it reacts to an action and produces something that needs to be interpreted. Integrating such software into teaching requires a rethinking of the learning tasks and teaching strategies. Thus, we decided to develop *mathematical learning units with computer algebra* to accompany the syllabus for high school mathematics (we use *Derive*).

One of our studies dealt with a unit for Grade 8 "Equations and Problems" (Zehavi & Mann, 1999). We discuss it here briefly in terms of Chevallard's notions. In traditional practice, one of the most difficult tasks for mathematics teachers is to teach the students how to translate word problems into equations (modeling). Within a CAS environment new tasks emerge, for example, students are asked to invent problems for a given type of model. Another emerging task is the manipulation of parametric equations as an integral part of introducing word problems. To accomplish these tasks the traditional technique of demonstration by the teacher and repetition of similar problems by the pupils is replaced by the following techniques:

- Teachers transfer the control over the modeling of given problems to the pupils. This can be achieved because in a CAS environment, creating the model by the student and solving it by CAS are inseparable. Moreover, the pupils are challenged to take responsibility for the appropriateness of the problems that they themselves invent.
- Computerized tutorials are designed to elicit mathematical ideas in the context of a given model. CAS enables the creation of didactic situations that lead young students, naturally, to the

notion of a parametric solution. The individual work of the pupils is followed by class discussion, thereby creating Webs of meanings (Noss & Hoyles, 1996). Consequently, the students learn about the constraints of a family of word problems.

The analysis and studies related to such new learning units prompted us to propose some initial elements of what we believe to be a new *praxeology* of teaching and learning mathematics with CAS.

A proposed scheme for teaching and learning mathematics with CAS

Based on Chevallard's ideas and our accumulated experience, we suggest a scheme for teaching a mathematical topic that comprises three components: *Types of tasks, Teaching techniques, and Principles.* We think that it is too early, however, to establish a theory of the functioning of mathematical knowledge in learning with computers, and therefore prefer to discuss the principles underlying the conceptual basis of a theory.

Types of tasks

- Tasks that are designed to overcome the identified difficulties in a specific topic. We chose to treat those difficulties for which we believe that the technology has the potential to help students to better understand the topic.
- New tasks that highlight mathematical ideas within a learned topic. Such tasks could not be dealt with practically or effectively without CAS.
- Tasks that extend the topic at hand by connecting it to previously learned topics or to topics that will be learned at a later stage.

Teaching techniques

- The learning activities utilize the technological tools to create special effects that motivate student interest and thinking.
- A sequence of activities starts with assignments designed to achieve specific goals for which the individual student is responsible, and gradually turns into open-ended assignments for which the group and the teacher share the responsibility.
- The teachers decide on the nature of their role on the basis of the following components: the cognitive aspects of learning which are revealed by observing individual student work, the interaction between the learners in the participants, and the use of specially designed tools for evaluation.

Principles

- When the merger of student work, CAS performance, and student reflection occurs – the 'mathematical assistant' (as *Derive* is called) serves as a learning environment.
- The teaching techniques need to consider both the up-to-date technology and modern approaches to student learning (e.g., the balance between conveying of knowledge by the teacher and its construction by the student).
- Since the role of the teacher who teaches with modern technology is so complex, the structure and options of a computer-based learning unit should be initially clear to teacher.

The suggested scheme was implemented in a formative study of a new unit. In the following we describe the design and development of the tasks and teaching techniques for this unit.

The design of a learning unit: Two Windows and One Parabola

Our aim was to develop a unit for Grade 10 that deals with parabolas. According to our scheme, we started by defining the types of tasks. The teachers, who participated in an in-service MathComp course, helped us to identify two main limitations of the traditional presentation of parabolas in the high school curriculum.

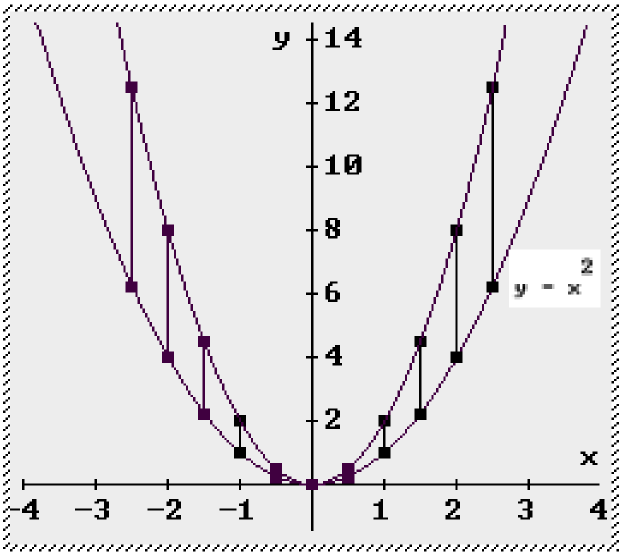
1. The difficulties students have in expressing the relationship between the parameter '*a* ' of the quadratic function $y = ax^2$ and the shape of its graph. Most of the students can determine when the parabola has a cup/hat shape. Students also say something about 'stretching/shrinking' parabolas, or that '*a* ' is the 'slope' of the parabola (analogous to a line).
2. The gap between the two viewpoints on the parabola in the traditional curriculum: the algebraic view of the graph of the quadratic equation, and the analytic-geometry view of loci.

To this end, we designed the following activities to aid students to better express the features of the shape of parabolas than they ordinarily do. These activities utilize the technological tools to create visual effects and prompt mathematical insight.

Activity I

One) Fill in the blanks in the given expression and reconstruct the shown figure (Figure 1).

Two) Modify the figure for other functions $y = ax^2$ ($a \neq 0$).



$$\text{VECTOR}\left(\begin{bmatrix} x & ? \\ x & ? \end{bmatrix}, x, ?, ?, ?\right)$$

Figure 1. Stretching/shrinking a parabola

Activity II

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One) Fill in the blanks in the given expression and reconstruct the
   shown figure (Figure 2).
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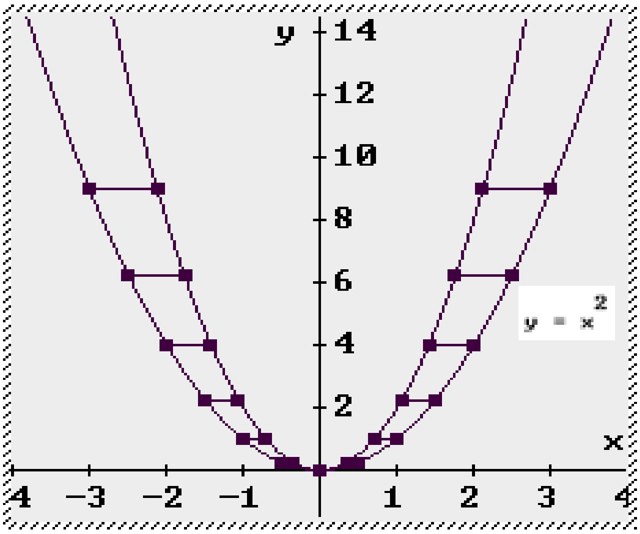
`Two) Modify the figure for other functions` $y = ax^2$.



$$\mathrm{VECTOR}\left(\begin{bmatrix} x & x^2 \\ ? & ? \end{bmatrix}, x, ?, ?, ?\right)$$

Figure 2. Opening/closing a parabola

Activity III

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One) Find the slope at points on the two parabolas.
Two) Can you find on each graph points at which the slope is 1 or –
   1?
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Our aim was twofold: to integrate the findings from these activities and to lead the way to viewing the parabola as a locus of points. Naturally, we concentrated on the *Focus point* of the parabola. Consequently, we developed the notion of "a special chord in a parabola" (perpendicular to the line of symmetry) (Figure 3).

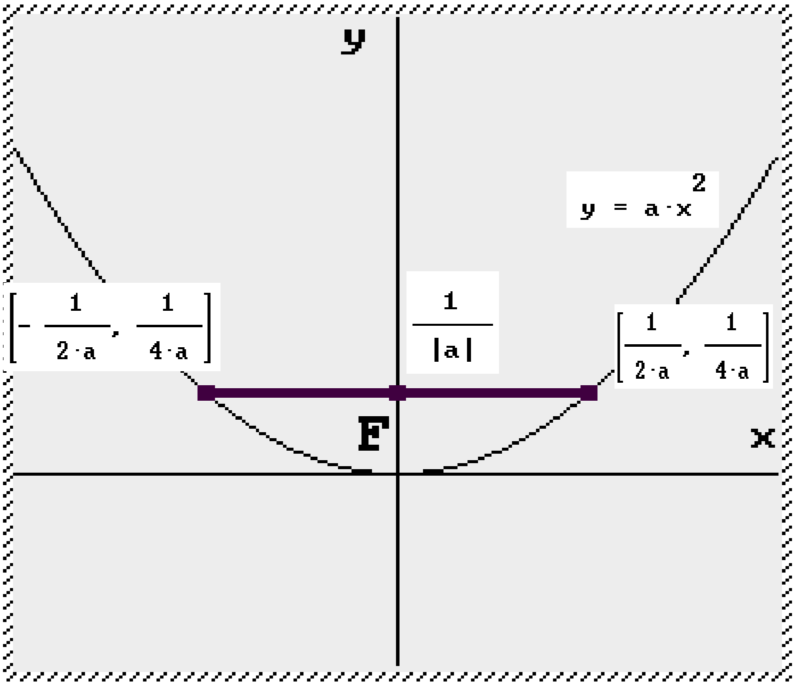

Figure 3. A special chord

The chord, which passes through the Focus point F$(0,\frac{1}{4a})$, has special properties. The length of this chord is $\frac{1}{a}$, which agrees visually with the fact that as '*a*' increases, the parabola 'shrinks'. When we calculate the distance between F and any point on the parabola $P(m, am^2)$, using our 'talented mathematical assistant', we get the expression $|P - F| = m^2 |a| + \frac{1}{4|a|}$. This expression, as we know, has a special geometric meaning. In addition, the slopes at the end-points of the chord are 1 or –1. Our task now is to suggest didactical strategies to introduce these ideas in the classroom.

Comparing two alternative practices

The team members and the teachers with whom we shared the basic ideas of the unit "Two windows and one parabola" were enthusiastic about the innovative approach. However, we could not agree on the sequence of activities. Thus, we decided to prepare two pilot versions and to compare the alternative approaches in class.

In the first version a straightforward definition that reveals the relation between the algebraic structure and the visual shape was given: " A special chord of the parabola $y = ax^2$ is the chord that connects two symmetric points on the parabola's arms, where the distance between the points is $\frac{1}{|a|}$." In the second version an analytic definition was given that should lead to the visual property: "A special chord of a parabola $y = ax^2$ is the chord that connects two symmetric points on the parabola's arms, where the slopes are 1 or –1." These definitions were followed by a sequence of activities that led to the description of the parabola as a collection of points with a specific property; the distance between a point P and the Focus point F is equal to the sum of the Y- coordinates of P and F. The first version concluded by finding the slopes at the end-points of the special chord. In the second version the length of the special chord was calculated. While many teachers preferred the second version, which seemed to them as 'more deductive', the 'more natural' first version appealed to other teachers.

The study

Two teachers volunteered to implement their preferred version of the unit in their Grade 10 classes. Class I (n = 25) used the first version, and Class II (n = 32) used the second version. Each class worked on the unit for 5 periods (one period on activities I-III, 3 periods on the sequence of activities dealing with the special chord and 1 period on three investigative problems that were derived from the previous parts). Members of the MathComp team joined the teachers in all the periods. In the following we shall first discuss the main findings in both classes for the two versions. Then, we shall compare students' work on the three investigative problems.

It became apparent that students in both classes were encountering conceptual difficulties. Although the students in class I performed the activities quite happily, it was observed that 10 of them (40%) felt at a certain stage that they just discovered that the length of the special chord is $\frac{1}{|a|}$, by this showing no understanding of the given definition. In class II the special chord was defined using the notion of slope. We realized that the students needed much help because they did not adequately master the concepts of slope and derivative. In spite of the difficulties the activities were completed in both classes and discussions were held. In the fifth period all the students worked on three problems.

Problem 1

One) Plot the graph of $y = 0.5x^2$ and its special chord [[,],[,]].

Two) Multiply by 2 the coordinates of the end-points of the special chord and plot the new segment 2*[[,],[,]].
Plot a parabola ______ where the new segment is its special chord.

Three) By what number do we have to multiply the coordinates of the end-points of the special chord in $y = 0.5x^2$ to obtain the special chord of $y = -1.5x^2$.

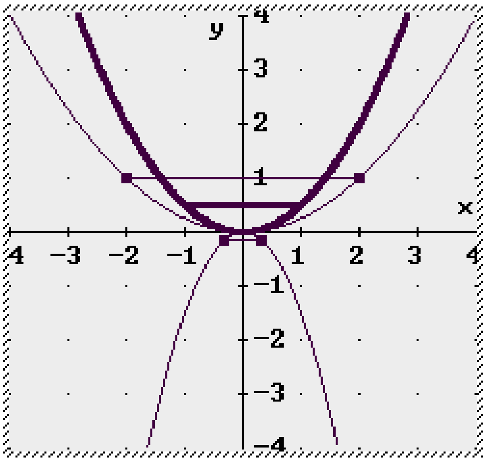

Figure 4. Fitting special chords
(The problem was presented without the figure.)

The students in class I completed the tasks within 5 minutes. Since they could easily check their answers by drawing, they got the correct answers. Moreover, 8 students, on their own initiative, reflected on their work and came up with the generalization: "If you want to fit the special chord of a parabola $y = ax^2$ to the parabola $y = bx^2$, you should multiply the coordinates by *a/b.*" In class II it took about 12 minutes to get the correct answers. We concluded that the first version probably helped better to conceptualize the 'reciprocal' relation between '*a*' and the shape of the parabola.

Problem 2

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The end-points of the special chord in a 'smiling' parabola are
(-1.5, 0.75) (1.5, 0.75).
Suggest at least two methods to determine which of the points
A(2, 2.25) B(1.5, 0.75) C(-3, 3) D(-4, 3.75) belong to the
parabola.
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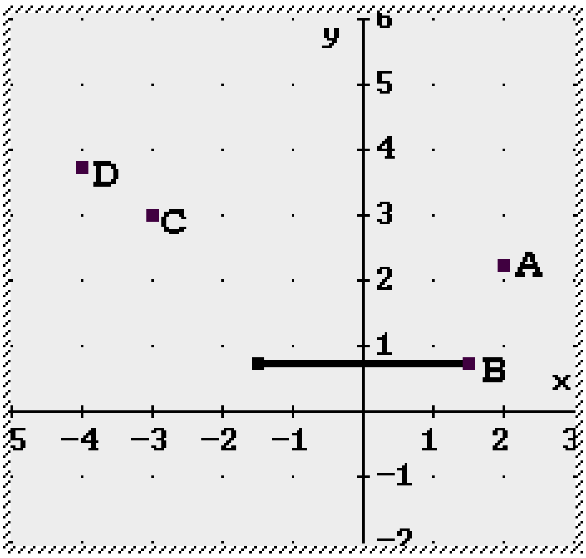

Figure 5. Fitting points
(The problem was presented without the figure.)

In class I, 19 students (76%) answered correctly; most of them wrote that they have two methods, algebraic and graphic (which were not actually different methods). They realized that the length of the special chord is 3, and concluded that '*a*' must be 1/3. Many jumped to the conclusion that the equation is $y = \frac{1}{3}x^2$, but only a few students justified explicitly their conclusion by comparing $\frac{1}{4a}$ with 0.75. Then they substituted the given coordinates in the equation. For the additional method, they plotted the graph and used the 'trace' mode to check which of the given points is on the graph. In the second class all the students gave correct answers. About 50% of them, in addition to the method described, came up with a method that utilized the property of the distance between points on the parabola and the mid-point of the special chord. Some of them even commented that if the distance between a given point and the mid-point of the special chord is more/less of the sum of the *y*-values, then the point is outside/inside the parabola, respectively. The teacher encouraged the students to state a general theorem and prove it geometrically. By doing so the class linked the algebraic and geometric methods.

Problem 3

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Look at Figure 6.
One) Find a parabola (a>0)for which the segment
  [[2, 0],[4,0]] is the special chord.
Two) Check whether the 'distance' property and the 'slope' property
   exist for the parabola you found.
Three)     Find a parabola where the segment
  [[0, -1],[0, 1]] is the special chord.
Four) Check whether the 'distance' property and the 'slope' property
   exist for the parabola you found.
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In both classes the teachers invited the whole class to work on the problem together, and challenged the students to come up with ideas. The teacher in class I did not provide the students with hints. The students started by translating the horizontal segment and obtained the parabola $y = 0.5x^2$, then subtracted 0.5 and later substituted '$x - 3$' for x. Some students said that there was no need to check distances and slopes because "it is the same parabola, only translated". Others were content to check these properties. They did not have time to work on the second task.

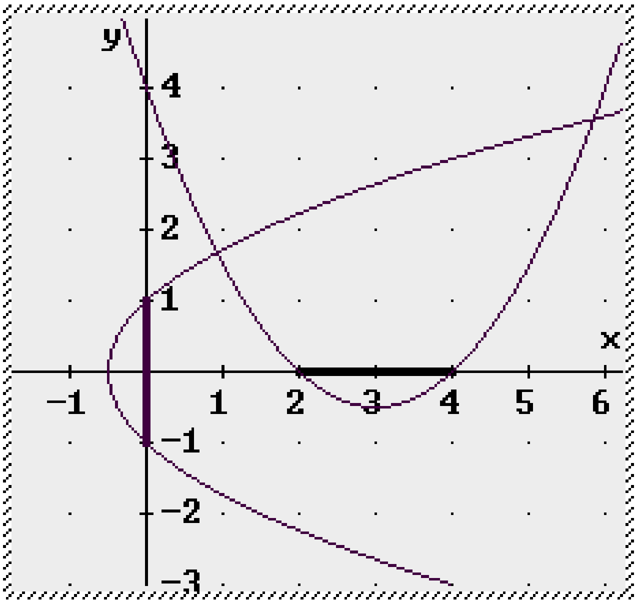

Figure 6. Fitting a parabola to a special chord

The teacher in the second class instructed the students to consider the intersection points with the *x*-axes. Thus, most of the students wrote the quadratic equation $y=0.5(x-2)(x-4)$, and used the software to plot the graph. Then, they were pleased when the 'computer' confirmed the 'distance' and 'slope' properties. The teacher did not provide any hint for the second case. The students plotted the graph of $x = -0.5y^2$ and translated it. Similarly to the first case, some students considered the *y* intercepts and obtained the equation $x = 0.5(y-1)(y+1)$. They mentioned the concept of inverse function (actually, inverse relation). Because of teacher intervention we could not attribute the different outcomes in the classes to the different versions they used.

Towards a theory of practices

The math education community awaited the recently published *Principles and Standards for School Mathematics* (NCTM, 2000). Those of us who are interested in using modern technologies to improve mathematics teaching and learning agree, of course, with the Technology Principle, which in essence says: (a) Technology influences the mathematics that is taught, and (b) Technology cannot replace the mathematics teacher.

The accumulated, integrated experience of practitioners and researchers can help to transform the NCTM blueprint into practice. While developing the proposed scheme for teaching and learning mathematics with CAS, we suggested tasks and teaching techniques to the teachers, and they in turn used them in their own classrooms. On the basis of the implementation we formulated our principles to be used as a basis of reasoning and a guide for teachers.

In conclusion we relate the findings of the study described in terms of the three principles in our scheme.

- CAS as a learning environment

Comparing alternative approaches was insightful. The findings for problem 1 show that the merger of student work, CAS performance and student reflection occurred only in the class that used the first version. However, for problem 2 the merger occurred only in the class that used the second version.

- Teaching techniques

Observing student work on the main sequence of activities (in periods 2-4) revealed difficulties of the teaching techniques in both versions. We are currently examining the findings to revise the unit.

- The role of the teacher

We could not compare the outcome of problem 3 in the two classes because of the different intervention of the teachers. It is also important to emphasize that the teacher in class II took advantage of the students' comments on problem 2 and invited the class to 'open a window' to synthetic geometry for justifying the hypotheses. On the other hand, the teacher of class I and the team member, did not use the pedagogical opportunity for problem 1. They did not explore the rule that the students discovered: "If you want to fit the special chord of the parabola $y = ax^2$ to the parabola $y = bx^2$ you should multiply the coordinates by *a/b*." The general mathematical concept that explains the students' rule is that mapping $[x, y]$ $[t*x, t*y]$is an expansion by the factor of 't' which is similarity mapping. The meaning of $[x', y'] = [t*x, t*y]$ is that $[x', y']$ is the image of $[x, y]$ by the radial expansion mapping with factor '*t*'. Figure 7 shows that when '*2x*' is substituted for '*x*' and '*2y*' is substituted for '*y*' in $y = x^2$, then the graph of the resulting equation, $y = 2x^2$, is the contraction of the original graph by a factor of ½. As a result of this episode we improved mathematically and didactically the teacher-training program for this unit.

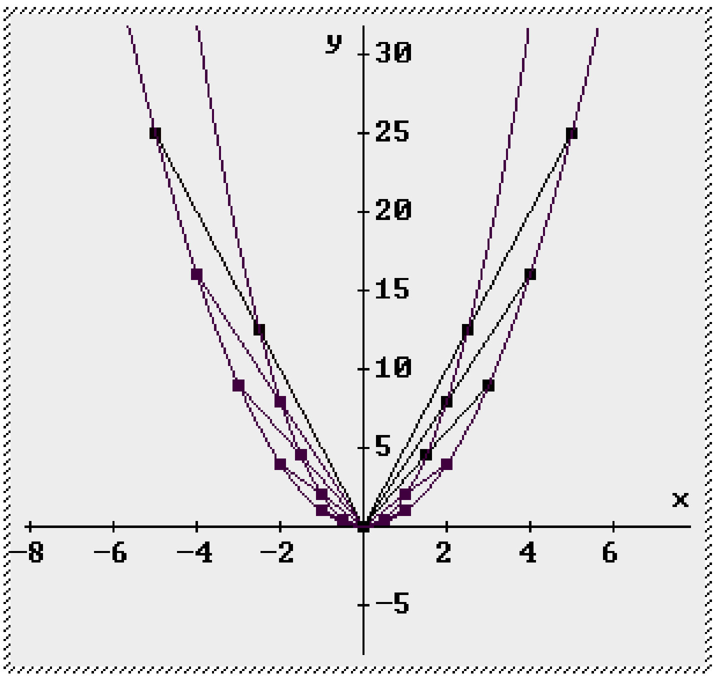

Figure 7. Expanding/contracting a parabola

Clearly, CAS technology challenges educators to design new tasks for students, new teaching techniques, and new ways to communicate them to teachers (Zehavi, 1997). With the ongoing research and implementation of the MathComp project, we intend to combine the continuous development of the *praxeology* with the formative development of new learning units.

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