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Fourth International Derive TI-89/92 Conference
Liverpool John Moores University, July 12 – 15, 2000

How Dynamic Geometry Systems could Complement Computer Algebra Systems (Linking Investigations in Geometry to Automated Theorem Proving)[1]

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Abstract

Two lines exist within Mathematical Software that have evolved independently: Computer Algebra Systems (CASs), specialised in exact and algebraic calculi, and Dynamic Geometry Systems (DGSs), specialised in "rule and compass Geometry". In this paper how these two kind of packages could evolve to complement each other is analysed. Following these ideas it would be possible to investigate with the DGS and then, if the result seems to be true, move to the CAS to obtain automatically a proof of it or to detect the exceptional cases when the theorem fails.

1. Introduction

Two lines exist within Mathematical Software that have evolved independently:

- Computer Algebra Systems (CASs), such as Maple, Derive, Mathematica, Axiom, Macsyma, Reduce, MuPad...
- Dynamic Geometry Systems (DGSs), such as The Geometer's Sketchpad, Cabri Geometry II, Cinderella, Euklid...

One of the main characteristics of CASs is that they use Exact Arithmetic (Fig. 1):

- Rationals are treated exactly, as fractions, without substituting them by their approximations in floating point arithmetic. E.g. *1/6+1/10=4/15* (Fig. 1).
- Reals are treated exactly too. For example: $(1/7)^{(1/5)}$, *ï6-ï3ï2=0*, ^... (Fig.1).
- Very big and very small numbers are not rounded. For instance: *300!* (Fig. 1).

The other main characteristic of CASs is that they can handle non-assigned variables and can automatically perform expansions and simplifications, e.g.

$$(x+y)^2-(x-y)^2=4xy$$

(without assigning a numerical value to x and y) (Fig. 1). This enables them to deal with polynomial expressions, or even more general symbolic expressions.

Symbolic differentiation and integration, linear and non-linear polynomial system solving, differential equations solving... are standard extensions of CASs (Fig. 1).

Also linear and non-linear equation and polynomial systems solving is included (Fig. 1). 2D and 3D plotting capabilities are usually included too. This allows for instance to interpret the solutions of a system of equations (Fig. 2).

[1] Partially supported by the DGES research project PB96-0098-C04-03.

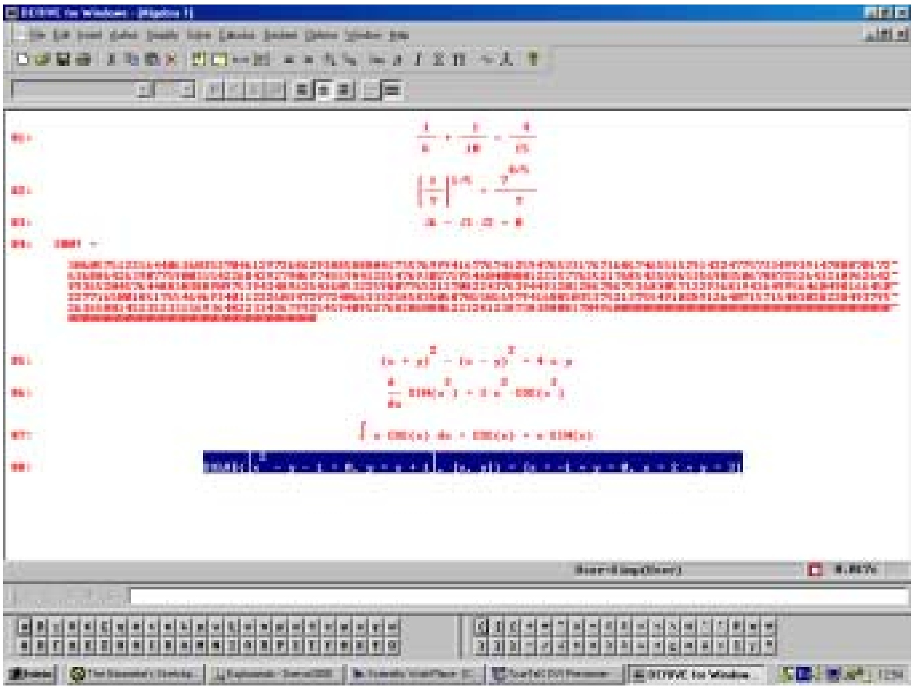

Fig.1: Some of the possibilities of the CAS Derive 5 [RRS,Ku].

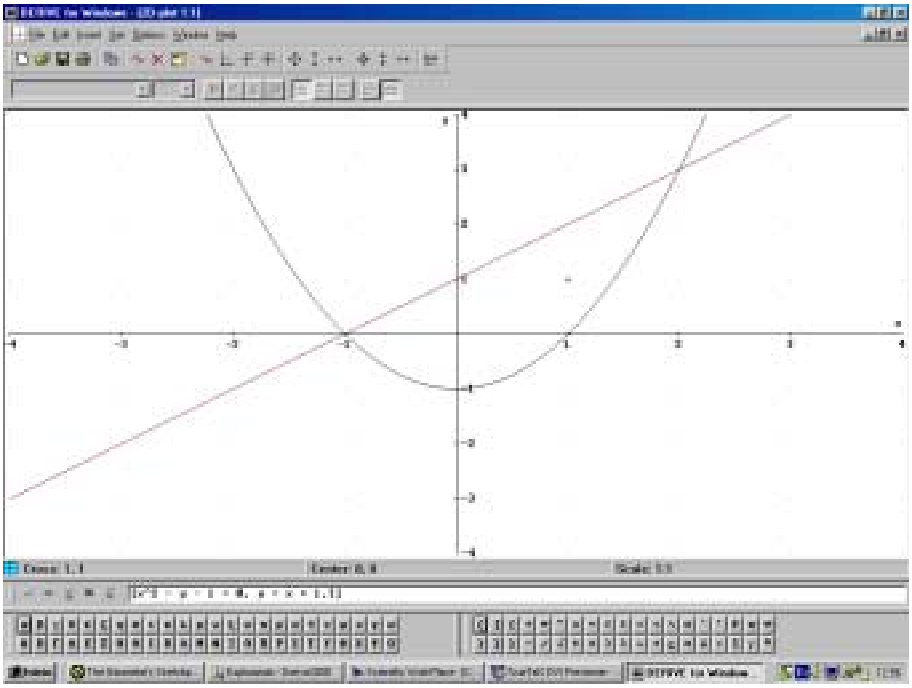

Fig. 2: Interpreting with Derive 5 the polynomial system of #8 in Fig. 1.

Meanwhile DGSs allow the mouse to perform the “ruler and compass’ Geometry” (Fig. 3). The adjective “dynamic” (and the importance of DGSs!) comes from the fact that, once a construction is finished, the first objects drawn (points)[2] can be dragged and dropped with the mouse, subsequently changing the whole construction (Fig. 4). They

[2] Denoted “parents” in The Geometer’s Sketchpad [An].

also incorporate very interesting gadgets like animation and tracing geometric loci (Fig. 5). This way it is possible to replicate other geometric processes like drawing an ellipse using the property that the sum of the distances to the foci is constant.

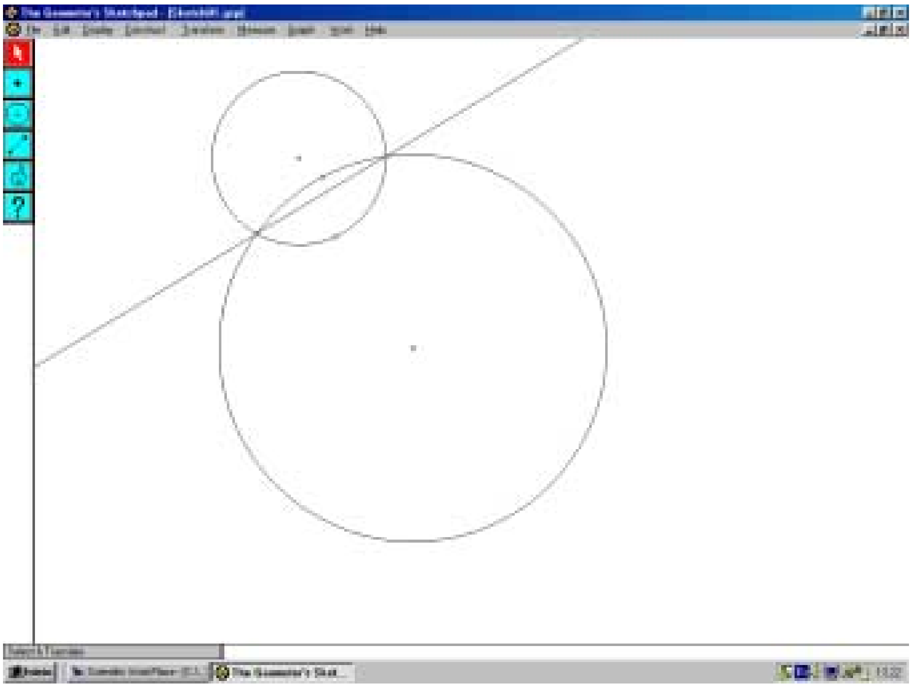

Fig. 3: Rule and compass' Geometry with The Geometer's Sketchpad 3.0.

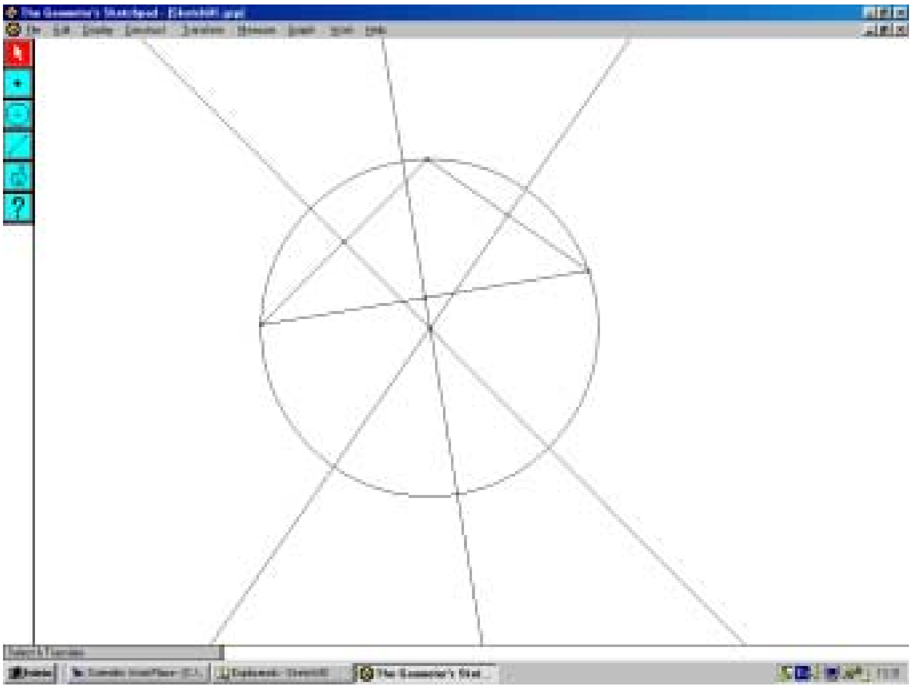

Fig. 4: Investigating with The Geometer's Sketchpad 3.0 if the circumcentre of a triangle is always in the interior of the triangle or not.

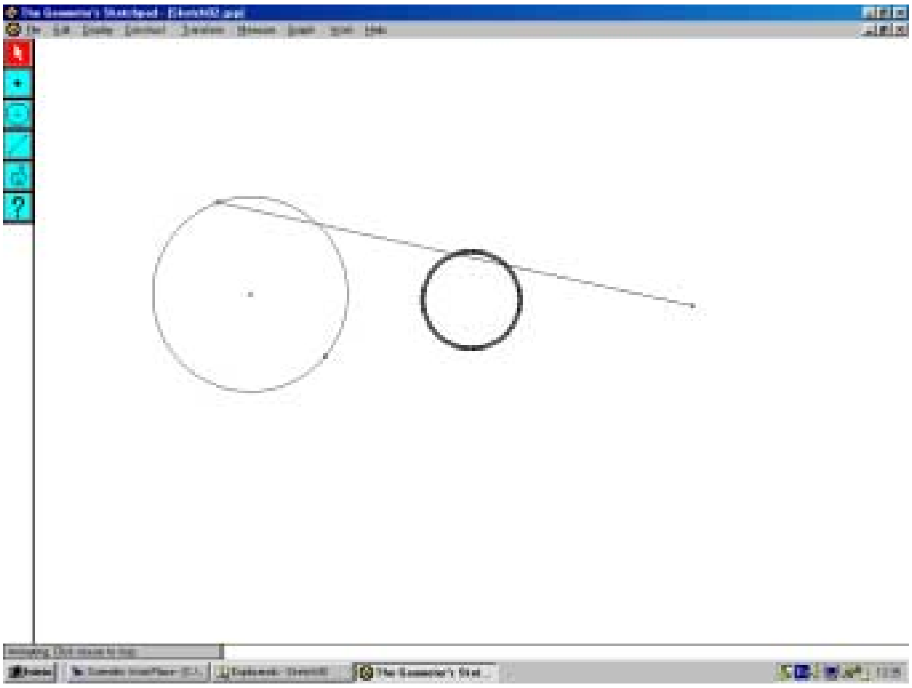

Fig. 5: Investigating with The Geometer's Sketchpad 3.0 the locus of the midpoint of a segment which endpoints are a fixed point and a point that lies on a circumference.

As far as we know (and we are related to the topic!: for instance the first author is Derive Beta-tester, Derive Consultant, Maple Ambassador for Spain, and has developed applications now incorporated to the regular versions of Maple, Macsyma and Derive), although some CASs (e.g. Maple [Cha]) include specific and powerful packages devoted to Euclidean Geometry (Fig. 6), no CAS has incorporated Dynamic

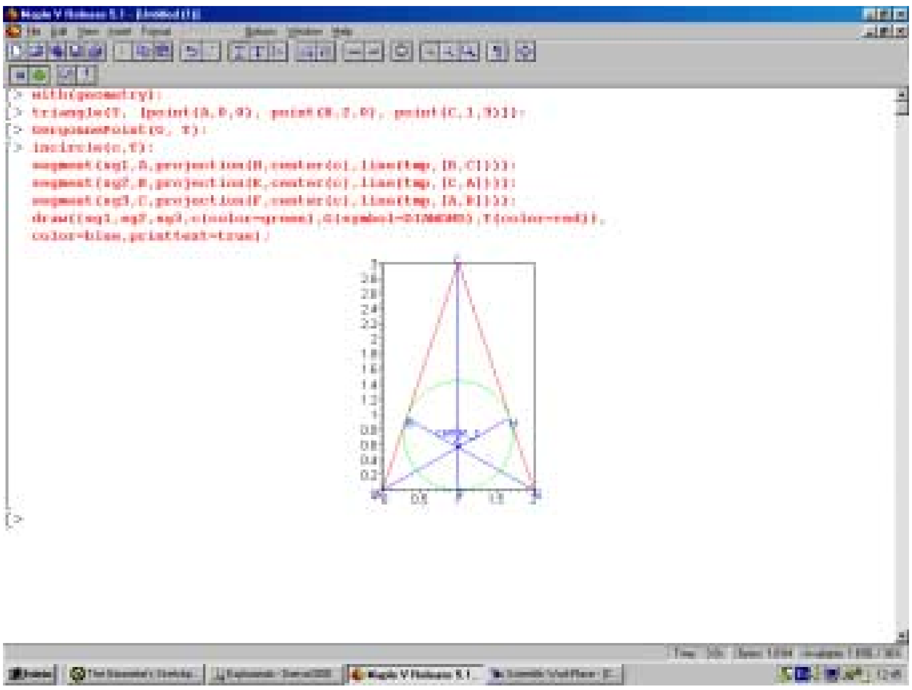

Fig. 6 Determining the Gergonne point of a triangle with Maple V's (Euclidean) Geometry package.

Geometry capabilities or, even more importantly, mouse-controlled capabilities (apart from those like selecting options for a plot or rotating a 3D graph).

On the other hand Dynamic Geometry Systems can't handle (at least from the point of view of the user) non-assigned variables. Therefore what can be saved from a DGS is “only”

- A “live” graphic (to be read by the DGS).
- A “geometric algorithm” (script or macro, to be interpreted by the DGS).
- A “dead” (fixed) graphic in one of the standard graphic formats (Postscript, bitmap, WMF...)
- “Numerical data” about the drawing. We mean by “numerical data” the (numerical) coordinates of a certain point or the equation of a certain line or circle related to some points with numerical coordinates (therefore these equations will necessarily have numerical coefficients!), length of objects...

  What DGSs don't offer for exporting info:
- “Parametric data” about the plot: coordinates of points (allowing parameters as coordinates), equations of objects (allowing parameters as coefficients), length of objects (depending on parameters). For example, if *A* is the intersection of axes *x* and *y*, its coordinates are *(0,0)*. The coordinates of a point *B* lying on the *x*-axis, can be in a certain moment *(2,0)*, but its parametric (general) coordinates are: *(b,0)*. Similarly, if C lies on the *y*-axis, its coordinates are *(0,c)* in general, although at a certain moment they can be e.g. *(0,1)*. Then the equation of line *BC* is *by+cx-bc=0* in general, despite the fact that in a certain moment it can be e.g. *2y+x-2=0*.
- Export equations to Derive, Maple, Mathematica... format.

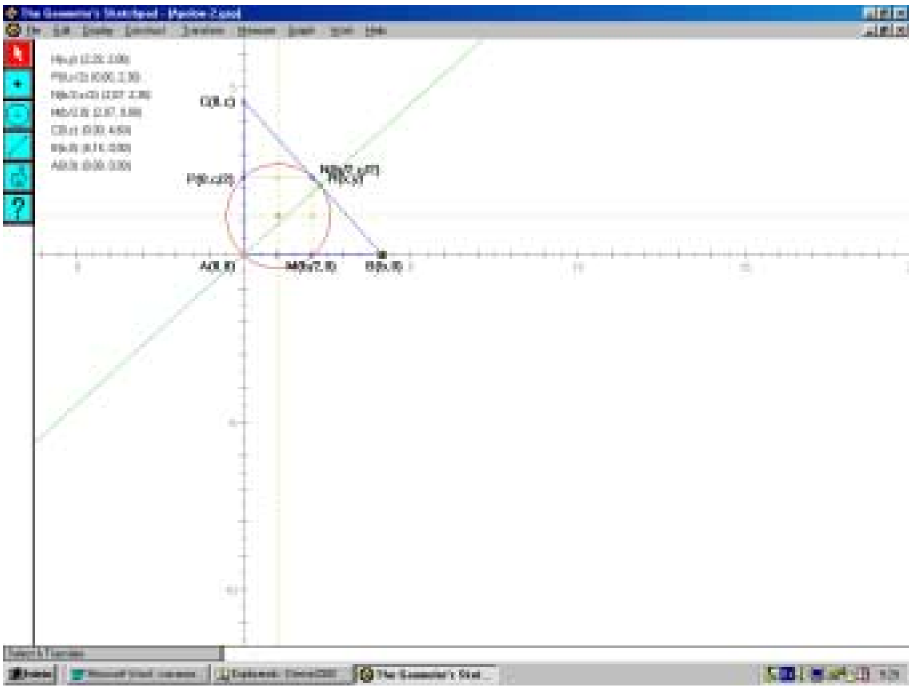

Fig.7: What should be possible with the CAS: parametric coordinates.

2. Some Ideas About Automatic Theorem Proving in Geometry

An obvious application of CASs is Automatic Theorem Proving in Geometry, using either Gröbner bases method [Bu,RR2] or Wu's pseudoremainder method [RR2,RR3, RR4,Wu1,Wu2]. Let us remark that these methods produce formal proofs from the mathematical point of view.

Example 1: Existence of orthocentre.

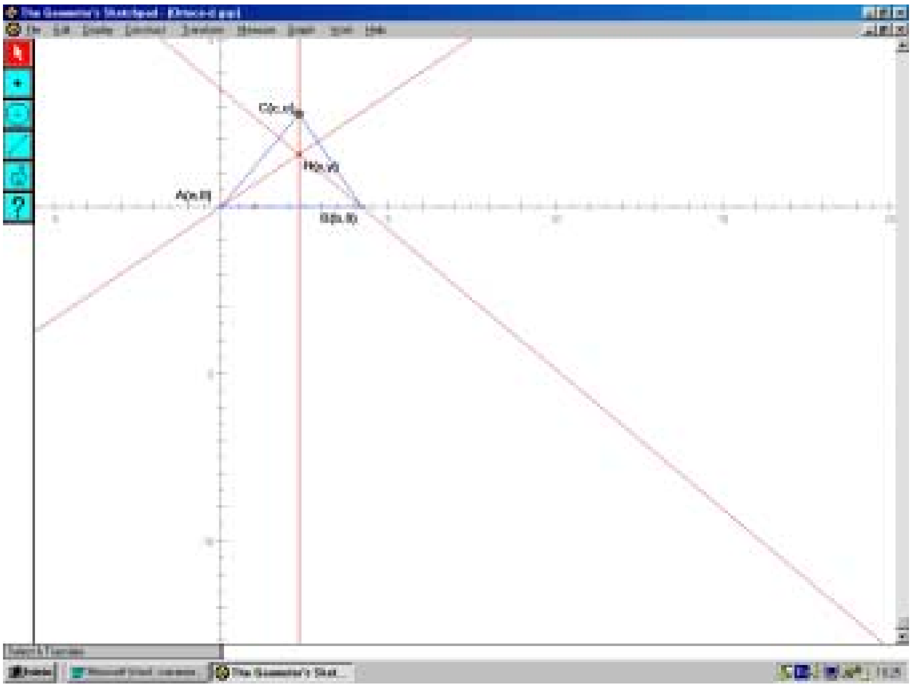

Fig. 8: Checking the existence of orthocentre with The Geometer's Sketchpad 3.0.

Proof (with Derive 5) of the existence of orthocentre:

```
#1: hyp1:=(c-b)*x+e*y=0

#2: hyp2:=c*(x-b)+e*y=0

#3: thes:=x-c=0

#4: SOLVE([hyp1,hyp2],[x,y])=[x=c AND y=c*(b-c)/e]

#5: SOLVE([hyp1,hyp2,thes],[x,y])=[x=c AND y=c*(b-c)/e]
```

Explanation: Equations *hyp1=0* and *hyp2=0* give the intersection point of two of the altitudes of a general triangle (located in a convenient way). If the equation of the third altitude, *thes=0*, is added, the system has the same solution. Therefore the three altitudes are concurrent.

Example 2: (Appolonius Theorem) Given a right triangle, the pedal point corresponding to the vertex where the angle is 90^0, belongs to the circle through the midpoints of the sides of the triangle.

Proof (with Derive 5) of Appolonius problem.

```
#1: hyp1:=c*x+b*y-b*c=0

#2: hyp2:=b*x-c*y=0

#3: thes:=x^2+y^2-b/2*x-c/2*y=0

#4: SOLVE([hyp1,hyp2],[x,y])=
               [x=b*c^2/(b^2+c^2) AND y=b^2*c/(b^2+c^2)]

#5: SOLVE([hyp1,hyp2,thes],[x,y])=
               [x=b*c^2/(b^2+c^2) AND y=b^2*c/(b^2+c^2)
                AND 0/=-b]
```

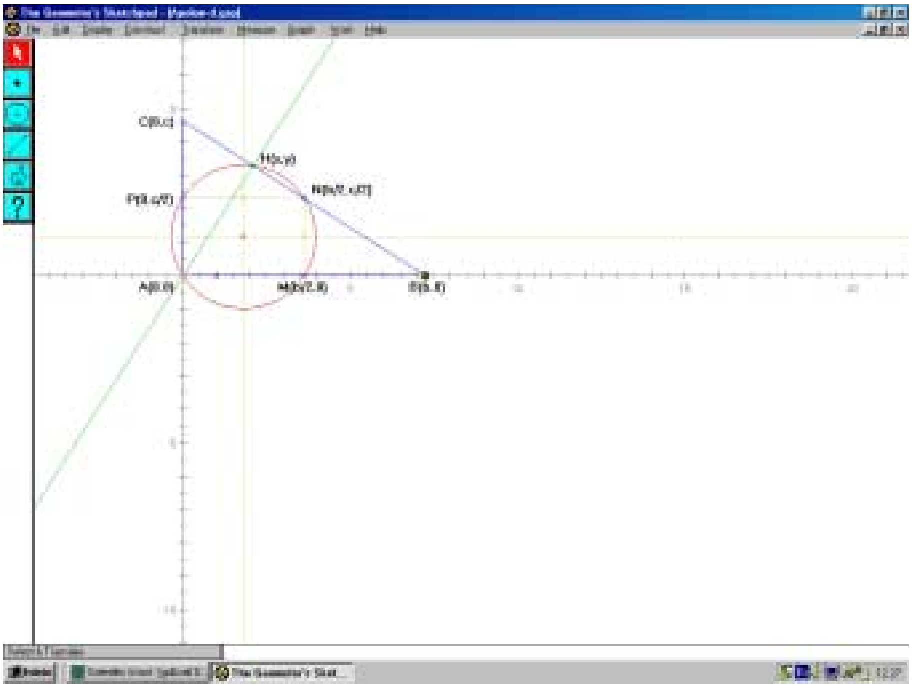

Fig. 9: Exploring Appolonius' Theorem with Sketchpad.

Explanation: Equations *hyp1=0* and *hyp2=0* give the pedal point of a general rectangle triangle (located in a convenient way). If the equation of the circle, *thes=0,* is added, the system has the same solution. Therefore the pedal point is on the circle. The extra condition $0 \neq b$ imposes that the triangle doesn't degenerate into a segment.

Remark: Unfortunately Derive 5 doesn't provide a command that calculates Gröbner bases (only a Gröbner bases-based "solve" command). This fact restricts the problems that can be proved with the Gröbner bases' method to simple cases as those shown above. Wu's method can be used without restrictions [RR3, RR4] just implementing a pseudoremainder function (using the remainder function provided)[RR1].

The algebraic theory can be found in the classics [AM,ZS,RM]. An excellent introduction to ideal theory and its applications (with a computational touch) is [CLO].

3. Complementing Each Other

So, a direct application of DGSs is investigating in Geometry, i.e. checking the validity of "guessed" results and even to try to discover (or rediscover) new results. Nice examples are the investigations in figures 4,5,8 and 9.

Meanwhile we are able to prove theorems as shown in examples 1 and the one mentioned in the previous paragraph.

Therefore, it would be very nice to link two of these applications. That way it would be possible to investigate with the DGS and then (if the result seems to be true) to move to the CAS to automatically obtain a proof of it or to detect the exceptional cases when the theorem fails.

4. Environment and Resume of the Process

[Step 1] With the GSD (or from another source of inspiration):
A geometric "guess" is established → Available.

[Step 2] With the GSD:
The thesis is tested (the configuration, including the hypotheses conditions is drawn and "dynamically" altered). As a conclusion the guess can be either rejected or considered "most probably" true → Available.

[Step 3] With the GSD:
The equations (with parametric coefficients) of the hypotheses and (possibly) thesis from the configuration are obtained → Has to be implemented by the designers of the GSD!

[Step 4] With the GSD (or through an external application):
The equations are translated from GSD format to the exact CAS format → Not-Available (should be included as an option of Step 3, but could be easily implemented as an external application)[3].

[Step 5] With the CAS:
An algebraic translation of the geometric operators and relations (point ... is on line ..., lines ... and ... are parallel...) should be provided in order to express the thesis → Not-Available[4], but can be easily implemented.

[3] For instance from The Geometer's Sketchpad it is possible to set *Display/Preferences* to *Text Format* in order to cut and paste to a CAS, but there are missing symbols (such as the "*").
[4] They are available in some CASs like Maple V (within the "Geometry" package).

[Step 6] With the CAS:
The thesis is automatically proved (using either Gröbner bases method or Wu's pseudoremainder method) → Available.

5. Conclusions

We think this is a very interesting cooperation that, as far as we know, is not available yet (although it would be easy to provide and, in some cases, as in the TI-92, both technologies are simultaneously available).

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