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Embedding *DERIVE* into Traditional Mathematics Courses

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Abstract

For several years now, the author has actively promoted the integration of computer algebra into the teaching, learning and assessment involved in traditional mathematics courses. This has been achieved by the careful design of course materials showing how symbolic computation and relevant graphics images can positively assist the learning process. On occasions, this involves the use of bespoke, user-defined commands in well known areas such as differential and integral calculus. In more advanced courses, appropriate utility commands are used to investigate solutions of differential equations and further user-defined commands are employed in, for example, plotting piece-wise defined periodic functions. The latter is particularly useful when overlaying the Fourier series representation for such signal functions.

This paper will present examples of the above in addition to examples of examination questions reflecting this style of course delivery.

Introduction

This paper addresses the use of CAS (*DERIVE*) as an intrinsic part of courses previously delivered via purely traditional means to undergraduates following awards in subjects other than Mathematics. In the UK, we often refer to such courses as “service Mathematics” and the emphasis is less on theory and more on applications and techniques. Students enrolled on these service courses at Anglia Polytechnic University are normally registered for awards in e.g. Electronics, Audio Technology and the Applied and Social Sciences. The service courses, normally referred to as modules, run over one semester and are text book based. However, either at the beginning or during the course of the delivery of the module, students are issued with supplementary *DERIVE* based materials that correspond directly to the content of the text book on which the module is based. Moreover, students are informed that one of the aims of the module is to “marry” a traditional text book exposition with symbolic computational investigations. In lectures, this aim is facilitated by on-line use of *DERIVE* via a lap top and data projector. The lectures are essentially interactive involving both the use of the whiteboard and the projected computer generated image.

The paper will now present a selection of topics, in order of difficulty, giving examples of how *DERIVE* (v.4.11) is embedded into the teaching, learning and assessment of traditional Mathematics courses to non-mathematicians.

1. Integral Calculus

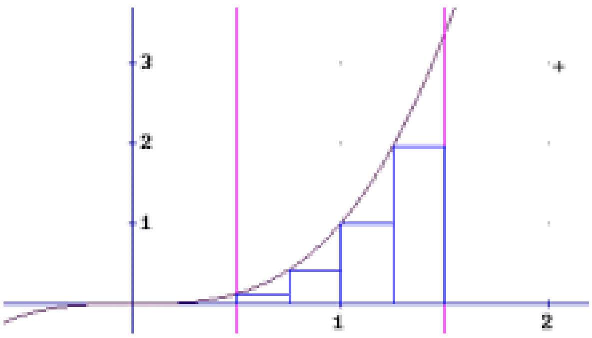
Before the advent of CAS, it was difficult to deal with the graphical aspects of definite integration associated with finding areas simply by using a whiteboard. Using *DERIVE*, it is relatively straightforward to design utility commands to split a prescribed area underneath a curve into a given number of rectangles and then to evaluate the sum of the component rectangular areas.

$$\text{DRAW\_RECTS}(u,\ x,\ a,\ b,\ n) := \text{VECTOR}\left(\lim_{x\to a}\begin{bmatrix} x+\dfrac{r\cdot(b-a)}{n} & 0 \\ x+\dfrac{r\cdot(b-a)}{n} & \displaystyle\lim_{x\to a+r\cdot(b-a)/n} u \\ x+\dfrac{(r+1)\cdot(b-a)}{n} & \displaystyle\lim_{x\to a+r\cdot(b-a)/n} u \\ x+\dfrac{(r+1)\cdot(b-a)}{n} & 0 \end{bmatrix},\ r,\ 0,\ n-1\right)$$

The above rather awesome looking command simplifies to a matrix of vectors each holding the co-ordinates of the corners of the *n* rectangles that approximate to the area bounded by the curve *u(x)*, the *x*-axis and the lines $x=a$ and $x=b$. The matrix of vectors (not shown here) can then be plotted. The definition would normally be hidden from the students but they would need to know how to interpret the arguments belonging to the command. As an example, we split the area bounded by the curve $u(x)=x^3$, the *x*-axis and the lines $x=0.5$ and $x=1.5$ into 4 rectangles. The required command is:

$$\text{DRAW\_RECTS}(x^3,\ x,\ 0.5,\ 1.5,\ 4)$$

The curve and the 4 rectangles are now plotted.



The next command simplifies to the sum of the areas of the *n* rectangles and has identical arguments to those of **DRAW_RECTS**.

$$\text{AREA\_RECTS}(u,\ x,\ a,\ b,\ n) := \frac{b-a}{n}\cdot\sum_{r=0}^{n-1}\ \lim_{x\to a+(b-a)\cdot r/n} u$$

Applying this command to our current example gives:

$$\text{AREA\_RECTS}(x^3, x, 0.5, 1.5, 4) = \frac{7}{8}$$

In some cases, as here, *DERIVE* will provide a closed form sum if the number of rectangles is not numerically specified i.e. remains as *n*.

$$\text{AREA\_RECTS}(x^3, x, 0.5, 1.5, n) = \frac{10 \cdot n^2 - 13 \cdot n + 4}{8 \cdot n^2}$$

It is now clear that the limiting value of the expression on the right as $n \to \infty$ is $\frac{5}{4}$. This can now be confirmed by direct integration:

$$\int_{0.5}^{1.5} x^3 \, dx = \frac{5}{4}$$

As a second example, consider finding the area bounded by the curve $u(x) = \frac{1}{x}$, the *x*-axis and the lines $x = 2$ and $x = 3$ first using 50 rectangles then a numerically unspecified number *n*.

$$\text{AREA\_RECTS}\left(\frac{1}{x}, x, 2, 3, 50\right)$$

$$0.407136$$

In this example, it is wise to use the approximate command/operator to evaluate the area, calculated here as 0.407136 sq. units.

$$\text{AREA\_RECTS}\left(\frac{1}{x}, x, 2, 3, n\right) = \sum_{r=0}^{n-1} \frac{1}{r + 2 \cdot n}$$

Note that *DERIVE* is unable to find a closed form for the sum in this example if *n* is not numerically specified. However, if we consider the sum to infinity we obtain the following!

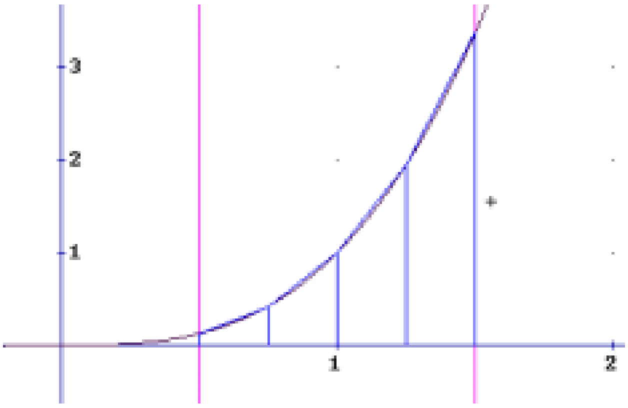
$$\lim_{n \to \infty} \text{AREA\_RECTS}\left(\frac{1}{x}, x, 2, 3, n\right) = \text{LN}\left(\frac{3}{2}\right)$$

Having constructed utility commands to draw and evaluate areas of rectangles, it is a fairly straightforward task to modify these commands to work with trapezia. We simply need to change the *y* co-ordinate of the upper right corner of each rectangle:

$$\mathrm{DRAW\_TRAPS}(u,\ x,\ a,\ b,\ n) := \mathrm{VECTOR}\left(\lim_{x \to a} \begin{bmatrix} x + \dfrac{r\cdot(b-a)}{n} & 0 \\ x + \dfrac{r\cdot(b-a)}{n} & \displaystyle\lim_{x \to a + r\cdot(b-a)/n} u \\ x + \dfrac{(r+1)\cdot(b-a)}{n} & \displaystyle\lim_{x \to a + (r+1)\cdot(b-a)/n} u \\ x + \dfrac{(r+1)\cdot(b-a)}{n} & 0 \end{bmatrix},\ r,\ 0,\ n-1\right)$$

We can now apply the command to our initial example:

$$\mathrm{DRAW\_TRAPS}(x^3,\ x,\ 0.5,\ 1.5,\ 4)$$



The accompanying area evaluation command becomes:

$$\mathrm{AREA\_TRAPS}(u,\ x,\ a,\ b,\ n) := \frac{b-a}{2\cdot n} \cdot \sum_{r=0}^{n-1} \left(\left(\lim_{x \to a + (b-a)\cdot r/n} u\right) + \lim_{x \to a + (b-a)\cdot(r+1)/n} u\right)$$

Applying the above command with an arbitrary number of trapezia gives a surprisingly simple closed form sum:

$$\mathrm{AREA\_TRAPS}(x^3,\ x,\ 0.5,\ 1.5,\ n) = \frac{5\cdot n^2 + 2}{4\cdot n^2}$$

As before, it is clear that the limiting value of the expression on the right as $n \to \infty$ is $\dfrac{5}{4}$.

2. Differential Equations

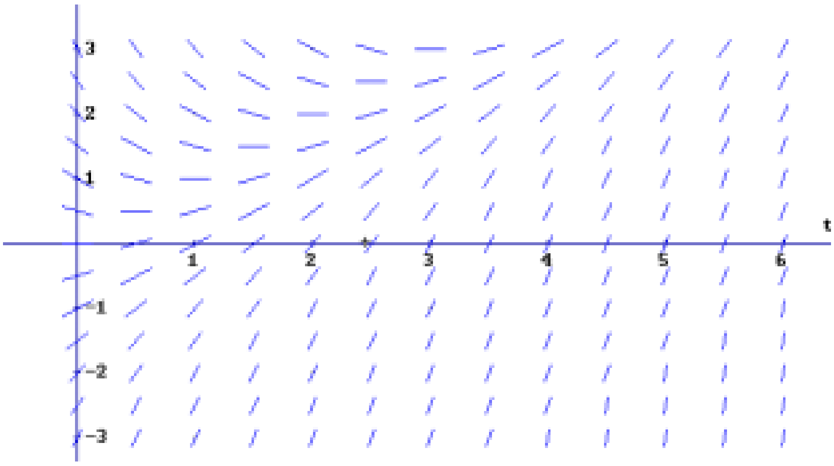
Students can be taught how to classify differential equations and, if analytically possible, to apply the appropriate technique in order to find the solution. However, with a tool such as *DERIVE*, much information can be obtained about the solution set without the need to solve a given differential equation. This is achieved by using *DERIVE* to plot the **direction or gradient field** associated with a differential equation.

As an example, consider $\frac{dy}{dt} = t - y$.

After having first loaded the ODE_APPR file, we can invoke the **DIRECTION_FIELD** command:

```
DIRECTION_FIELD(t - y, t, 0, 6, 12, y, -3, 3, 12)
```

This simplifies or approximates to a large matrix (not shown here) whose elements hold the co-ordinates of the end points of the tangent line segments associated with the differential equation. Plotting the matrix of co-ordinates shows:



This approach is a revelation for students who can now anticipate the nature of the solution set clearly indicated by the gradient field diagram. In particular, the diagram shows asymptotic behaviour towards the line with equation $y = t - 1$.

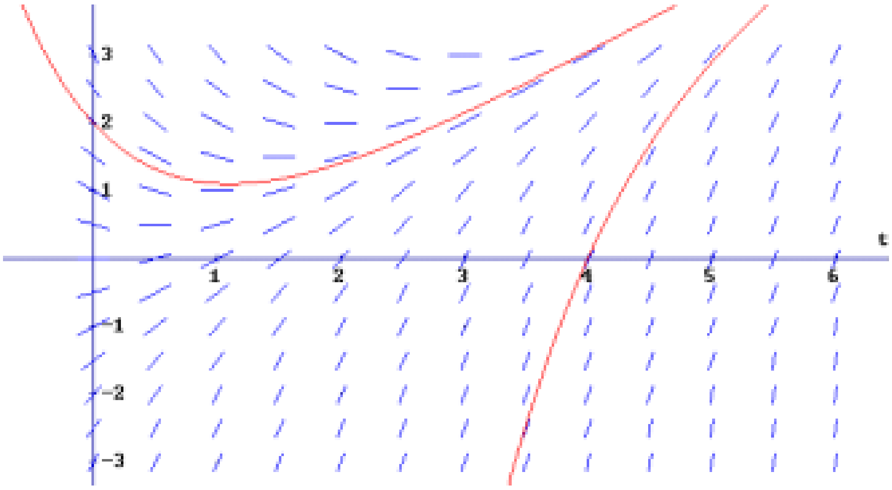
Solving this differential equation in the usual way yields:

$$y = t - 1 + ce^{-t}$$

In order to find particular solutions, we can choose to find those passing through say $(0,2)$ and $(0,4)$ obtaining:

$$y = 3e^{-t} + t - 1 \text{ and } y = -3e^{4-t} + t - 1 \text{ respectively.}$$

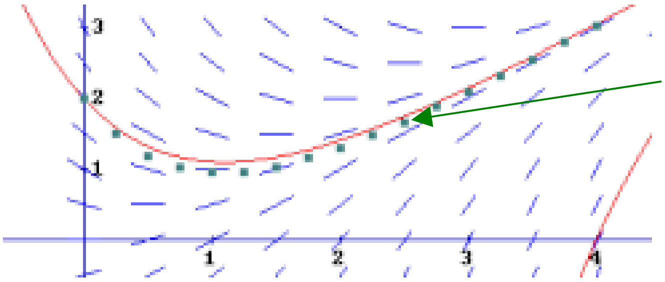
Superimposing these particular solutions onto the gradient field diagram shows:



An important point to emphasise to students is that, in practice, it may not be possible to solve a differential equation analytically. However, having obtained the gradient field diagram, we can nevertheless see what the solution set should look like. At this stage, it is appropriate to deal with simple numerical techniques for solving differential equations such as the Euler method. In order to assess the accuracy of the Euler technique, we can consider applying it to finding the particular solution of the previous differential equation passing through $(0, 2)$.

```
EULER(t - y, t, y, 0, 2, 0.25, 16)
```

The approximate operator produces a vector (not shown here) of 17 solution points beginning with $(0, 2)$ and incrementing t in steps of 0.25 until we reach $(4, 3.03006)$. A useful comparison with the actual solution curve can be seen by superimposing the approximate solution points onto the previous plot.



Solution points generated by the Euler command.

It is a simple matter to obtain a table of values showing the actual solution points by appropriately simplifying the command:

$$\text{VECTOR}\left(\left[t,\ 3\cdot\hat{e}^{-t} + t - 1\right],\ t,\ 0,\ 2,\ 0.25\right)$$

The students can then be informed that more accurate numerical techniques can be employed such as the classic fourth order Runge-Kutta method. When applied to the above example the appropriate command is:

```
RK([t - y], [t, y], [0, 2], 0.25, 16)
```

In summary, showing the first few points only, we have:

Euler		*Runge-Kutta*		*"Exact"*	
0	2	0	2	0	2
0.25	1.5	0.25	1.58642	0.25	1.58640
0.5	1.1875	0.5	1.31962	0.5	1.31959
0.75	1.01562	0.75	1.16714	0.75	1.16709
1	0.949218	1	1.10368	1	1.10363
1.25	0.961914	1.25	1.10955	1.25	1.10951
1.5	1.03393	1.5	1.16943	1.5	1.16939
1.75	1.15045	1.75	1.27135	1.75	1.27132
2	1.30033	2	1.40603	2	1.40600

Assessment Example

(June 1998 Examination paper)

Write down the DIRECTION_FIELD command in *DERIVE* which would be used to draw a gradient field for the general solution of the differential equation

$$\frac{dy}{dt}+ty=t$$

in the region $-3\le t\le 3$ and $-1\le y\le 3$ using 12 steps for both ranges.

This gradient field is shown in the diagram in the Appendix. Draw two or three suggested solution curves onto the diagram.

By separating the variables or otherwise, find the explicit solution of the differential equation given $y=3$ when $t=0$. Sketch this particular solution, suitably labeled, onto the gradient field diagram.

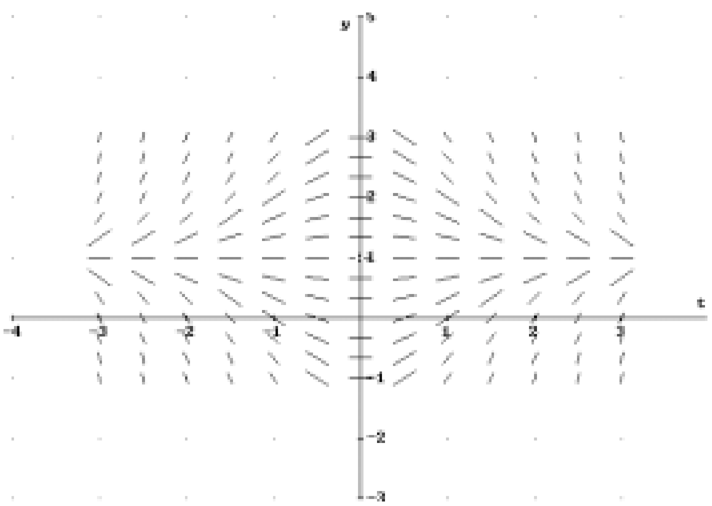


Diagram (reduced) provided in Appendix material for the examination.

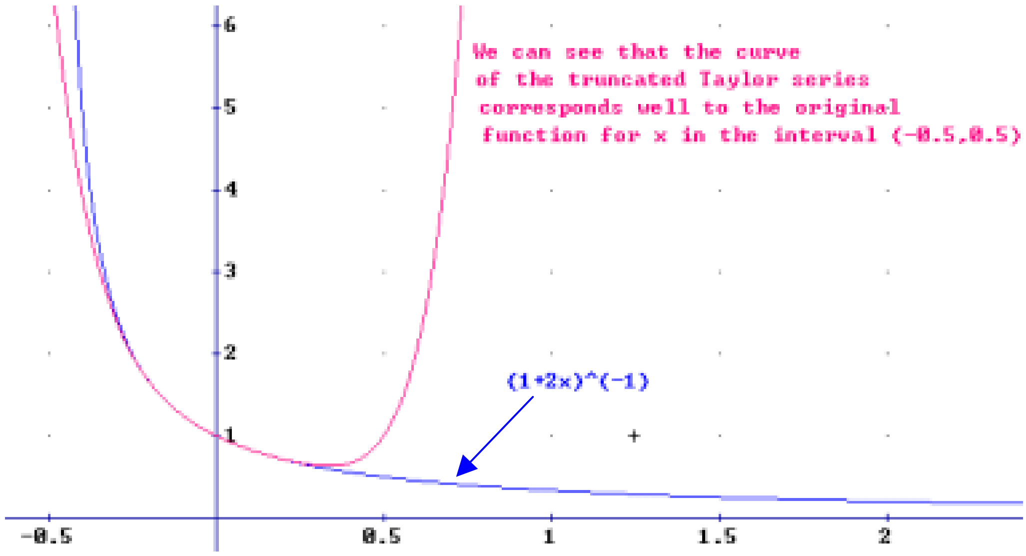
In addition, students are furnished with a sheet of relevant *DERIVE* utility commands.

3. Taylor and Fourier Series

3.1 Taylor Series

It is almost impossible to visually demonstrate the effect of plotting a truncated Taylor series of a function together with the graph of the original function, using only a white board. In the case of functions whose Taylor expansions are only valid for a constrained interval of the independent variable, *DERIVE* is invaluable in highlighting the range of validity in such cases.

As an example, consider the Taylor expansion of $(1+2x)^{-1}$ about 0 truncated just after the term involving x^6 i.e. $1-2x+4x^2-8x^3+16x^4-32x^5+64x^6$



The author has found that the construction of Taylor series provides a useful precursor for the more difficult topic of Fourier series. By the time the student meets Fourier series he/she should know that it may be possible to find an equivalent representation for suitable functions in terms of a series involving powers of the independent variable. A Fourier series does a similar job for suitable periodic functions except that the terms of the series are sines and/or cosines of the independent variable.

3.2 Fourier Series

We begin this section by seeing how *DERIVE* can be used to define and plot piecewise periodic functions. Consider the following such function:

$$f(t)=\begin{cases}-1, & -\pi<t<-\pi/2\\ 3, & -\pi/2\le t\le \pi/2\\ -1, & \pi/2<t<\pi\end{cases}$$

$$f(t+2\pi)=f(t)\ \text{ for all } t$$

This function is readily defined, over a fundamental period, in *DERIVE* using the CHI function as follows:

$$\mathrm{F(t)} := -1\cdot \mathrm{CHI}\left(-\pi,\ t,\ -\frac{\pi}{2}\right)+3\cdot \mathrm{CHI}\left(-\frac{\pi}{2},\ t,\ \frac{\pi}{2}\right)-1\cdot \mathrm{CHI}\left(\frac{\pi}{2},\ t,\ \pi\right)$$

The author has designed a command that results in a plot of any periodic function that has been previously defined over a fundamental period. This command, called **PERIODIC_PLOT**, has been deliberately designed to be compatible with the arguments of *DERIVE*'s **FOURIER** utility command.

$$\text{PERIODIC\_PLOT}(u,x,a,b,n):=\sum_{k=\text{FLOOR}(-(n-2)/2)}^{\text{FLOOR}(n/2)}\left(\text{STEP}(x-k\cdot b+a\cdot(k-1))\cdot\lim_{x\to x-k\cdot(b-a)}u-\text{STEP}(x+k\cdot a-b\cdot(k+1))\cdot\lim_{x\to x-k\cdot(b-a)}u\right)$$

When simplified, this command can be used to generate a plot of n cycles of the periodic function $u(x)$ defined on the periodic interval (a, b) . If n is odd, *DERIVE* will plot $\frac{n-1}{2}$ cycles on each side of the interval (a, b). If n is even, *DERIVE* will plot $\frac{n-2}{2}$ cycles to the left of (a, b) and $\frac{n}{2}$ cycles to the right of (a, b). This command is now used on the previously defined function $f(t)$.

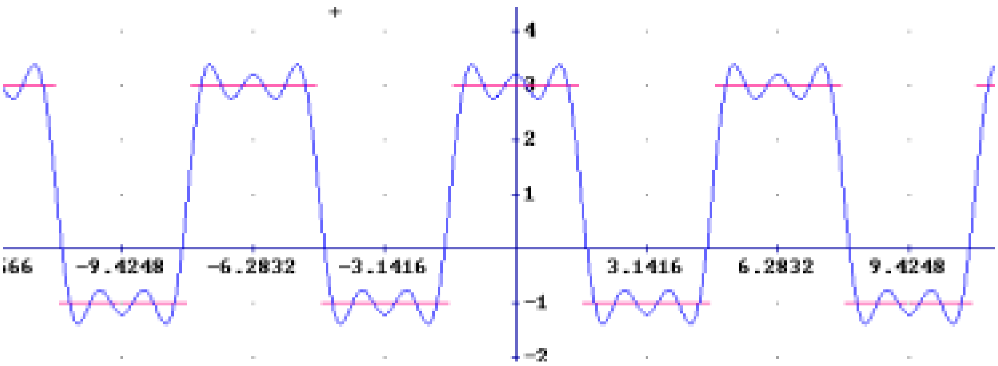
Simplifying this command (the simplification is not shown here) and plotting yields:

PERIODIC_PLOT(F(t), t, -π, π, 5)

We can now invoke the **FOURIER** command and thus superimpose say the first five harmonics of the Fourier series representation for *f(t)*.

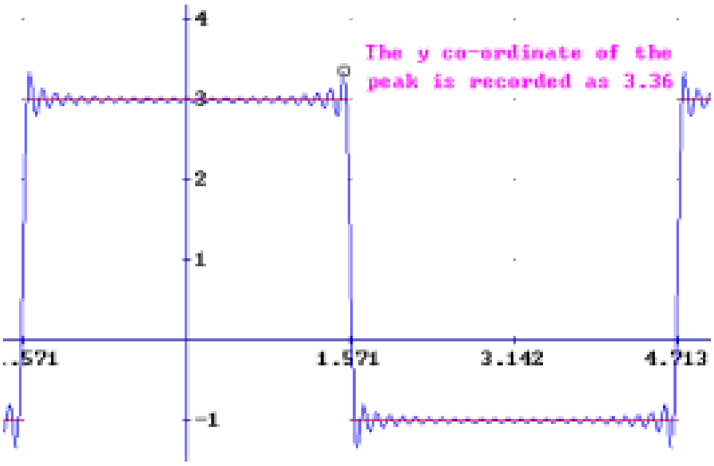
FOURIER(F(t), t, -π, π, 5)

$$\frac{8\cdot\cos(5\cdot t)}{5\cdot\pi}-\frac{8\cdot\cos(3\cdot t)}{3\cdot\pi}+\frac{8\cdot\cos(t)}{\pi}+1$$



As mentioned in previous examples, it would be very difficult to deal with this topic in the way shown above merely by using a whiteboard.
Plotting the first five harmonics already shows the onset of Gibbs' phenomenon, namely the "overshoot" immediately before and after a point of discontinuity. The literature states that at a

point of discontinuity, say where $x=\alpha$, the magnitude G of the total overshoot approaches 18% of $|f(\alpha-0)-f(\alpha+0)|$. Using *DERIVE*'s trace facility in the plot window, allows us to *measure* the overshoots and make comparisons with the result stated in the theory.



The above graph was obtained by generating, then plotting the first 50 harmonics of the Fourier series of our earlier example. We see that the total overshoot amounts to $2\times 0.36=0.72$ and that $|f(\pi/2-0)-f(\pi/2+0)|=4$. In this case we obtain precisely $\frac{0.72}{4}\times 100=18\%$!

Assessment Example

(January 1999 Examination paper)

A function $f(t)$ is defined in the following way:

$$f(t)=t, \qquad -\frac{\pi}{2}<t<\frac{\pi}{2}$$

$$f(t+\pi)=f(t) \qquad \text{for all } t$$

Sketch $f(t)$ between $t=-\pi$ and $t=2\pi$.

Explain, briefly, why the Fourier series representation for $f(t)$ is of the form

$b_1\sin\omega t+b_2\sin 2\omega t+b_3\sin 3\omega t+b_4\sin 4\omega t+\cdots$.

Line 1 in the following *DERIVE* extract shows the simplest form of the integral expression for the Fourier coefficients b_n, for $n\geq 1$.

```
#1:  4/π · ∫[0, π/2] t·SIN(2·n·t) dt

#2:  "The above integral simplifies to:"
```

$$\#3: \quad \frac{\mathrm{SIN}(\pi\cdot n)}{\pi\cdot n^2}-\frac{\mathrm{COS}(\pi\cdot n)}{n}$$

Using line 3 of this extract, write down the first four non-zero terms of the Fourier series representation of $f(t)$. By setting $t=\frac{\pi}{4}$, use these 4 terms to show that $f(\frac{\pi}{4})\approx\frac{2}{3}$.

DERIVE can now be used to synthesise $f(t)$ via n- term truncations of the Fourier series representation for $f(t)$. A function $synth(t,n)$ is defined (but not shown here) to be the Fourier series representation consisting of the first n terms. For example, we see from the extract below that $synth(\frac{\pi}{4},4)$ yields the same approximate value for $f(\frac{\pi}{4})$ as that given above. By comparison with the given waveform, comment briefly on the two results shown on lines 7 and 8.

(**Note**: 1.5549 is slightly less than $\frac{\pi}{2}$)

With reference to these results, use a sketch or otherwise to describe the behaviour of the synthesised waveform particularly around the points of discontinuity. Is the result shown on line 8 consistent with the 18% figure quoted in the literature?

#5: Precision := Approximate

#6: $\text{SYNTH}\left(\frac{\pi}{4}, 4\right) = 0.666666$

#7: $\text{SYNTH}\left(\frac{\pi}{4}, 100\right) = 0.780398$

#8: SYNTH(1.5549, 100) = 1.83560

Conclusion

The live use of *DERIVE* in lectures has been universally welcomed by students for several reasons.

- The easy to use graphics facilities allow the students to visualise and better understand concepts that are particularly difficult for non-specialist Mathematicians.
- *DERIVE* ***brings Mathematics to life***, allowing controlled experimentation and investigation to take place on the spot.
- Seeing *DERIVE* used in the classroom helps to remove any phobias the students may have about using the software in computer labs on their own.

The author is firmly of the view that both teaching and learning processes have been greatly enhanced by careful integration of *DERIVE* into traditional Mathematics courses.