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Generating Sturm Sequences With *Derive* and Applications

Robert J. Hill
Valparaiso University, USA
Email: robertjohill@home.com

Let $p(x)$ be a polynomial of degree n with real coefficients. We will generate the sequence of polynomials which we will call a Sturm sequence. Let $p_0(x) = p(x)$, $p_1(x) = p'(x)$

$$p_0(x) = q_1(x)\,p_1(x) - p_2(x)$$
$$p_1(x) = q_2(x)\,p_2(x) - p_3(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$p_{k-2}(x) = q_{k-1}(x)\,p_{k-1}(x) - p_k$$

It is noted that the degree of $p_i(x)$ is less than the degree of $p_{i-1}(x)$ and that the sign of the remainders are the negative of those in the Euclidean algorithm. The degree of $p_k(x)$ is zero and stops the process. If $p(x)$ has zeros of multiplicity greater than one then the last nonzero remainder is the greatest common divisor of $p(x)$ and $p'(x)$ which implies that $p(x)$ divided by the last nonzero remainder has only simple zeros which are the same as those of $p(x)$.

The basic *Derive* command which generates the Sturm sequence is

$$\left(\mathrm{ITERATES}\left(\left[v_2, -\ \mathrm{REMAINDER}(v_1, v_2)\right], v, \left[P(x), \frac{d}{dx}P(x)\right], n\right)\right)\downarrow\downarrow 1$$

As an example consider

$$P(x) := x^3 - \frac{63\cdot x^2}{10} + \frac{1223\cdot x}{100} - \frac{7161}{1000}$$

$$\left(\mathrm{ITERATES}\left(\left[v_2, -\ \mathrm{REMAINDER}(v_1, v_2)\right], v, \left[P(x), \frac{d}{dx}P(x)\right], 3\right)\right)\downarrow\downarrow 1$$

$$\left[x^3 - \frac{63\cdot x^2}{10} + \frac{1223\cdot x}{100} - \frac{7161}{1000},\ 3\cdot x^2 - \frac{63\cdot x}{5} + \frac{1223}{100},\ \frac{10\cdot x - 21}{15},\ 1\right]$$

It is convenient to write the basic Sturm generator as a function.

$$\text{STURM}(p,\ x,\ n) := \left(\text{ITERATES}\left(\left[v_2,\ -\ \text{REMAINDER}(v_1,\ v_2)\right],\ v,\ \left[p,\ \frac{d}{dx}\ p\right],\ n\right)\right)\downarrow\downarrow 1$$

Let $S(a)$ be the Sturm sequence evaluated at *a* and let $C(S(a))$ be the number of changes in sign of the elements of $S(x)$ when evaluated at *a*. Sturm's theorem states that if $p(x)$ has only simple zeros, then the number of real zeros between *a* and *b* $(a<b)$ is $C(S(a))-C(S(b))$. We note that if we compute $S(1)$ and $S(2)$ we obtain

$$S(1)$$

$$\left[-\frac{231}{1000},\ \frac{263}{100},\ -\frac{11}{15},\ 1\right]$$

$$S(2)$$

$$\left[\frac{99}{1000},\ -\frac{97}{100},\ -\frac{1}{15},\ 1\right]$$

and $C(S(1))=3$, $C(S(2))=2$. It is noted by Sturm’s theorem there is one real zero of $p(x)$ in $[1,2]$. The zero is actually $x=1.2$ as can be readily verified by Newton’s iteration.

We may count the number of sign changes in $S(a)$ by considering the sign of the product $S(a)_i\,S(a)_{i+1}$. If the product is negative the sign has changed and if the product is positive the sign has not changed. In fact the number of sign changes in $S(a)$ may be counted by

$$\sum_{i=1}^{n}\frac{-\text{SIGN}\left(S(a)_i\,S(a)_{i+1}\right)+1}{2}$$

provided no element of $S(a)$ is zero. It is convenient to automate the counting of sign changes and step through a range of values to isolate the zeros of $p(x)$. The *Derive* function that accomplishes this task is

$$\text{SEARCH}(x,\ n,\ l,\ u,\ \delta) :=$$

$$\text{VECTOR}\left(\left[S(x),\ x,\ \sum_{i=1}^{n}\frac{-\ \text{SIGN}((S(x))_i\cdot(S(x))_{i+1})+1}{2}\right],\ x,\ l,\ u,\ \delta\right)$$

The parameters required by this function are:

1. The variable along which we are searching x
2. The degree of the polynomial $p(x)$
3. The lowest value of x or the value where the search is to begin
4. The highest value of x or the value where the search is to end
5. The increment of the search

The output is a vector with 3 components.

1. A vector giving the Sturm sequence evaluated at x
2. The value x
3. The number of sign changes in the Sturm sequence when evaluated at x.

We will give some examples.

Example 1. Isolate the zeros of the polynomial

$$p(x) = x^4 - 6.28x^2 + 6.9696$$

$$P(x) := x^4 - 6.28 \cdot x^2 + 6.9696$$

$$\mathrm{STURM}(P(x),\ x,\ 4)$$

$$\left[x^4 - 6.28 \cdot x^2 + 6.9696,\ 4 \cdot x^3 - 12.56 \cdot x,\ 0.0008 \cdot (3925 \cdot x^2 - 8712),\ 3.68152 \cdot x,\ 6.9696\right]$$

$$S(x) := \left[x^4 - 6.28 \cdot x^2 + 6.9696,\ 4 \cdot x^3 - 12.56 \cdot x,\ 0.0008 \cdot (3925 \cdot x^2 - 8712),\ 3.68152 \cdot x,\ 6.9696\right]$$

$$\mathrm{SEARCH}(x,\ 4,\ -3,\ 3,\ 0.5)$$

$$\left[\begin{array}{ccc} [31.4496, -70.32, 21.2904, -11.0445, 6.9696] & -3 & 4 \\ [6.78209, -31.1, 12.6553, -9.20382, 6.9696] & -2.5 & 4 \\ [-2.1504, -6.88, 5.5904, -7.36305, 6.9696] & -2 & 3 \\ [-2.09789, 5.34, 0.0954, -5.52229, 6.9696] & -1.5 & 3 \\ [1.6896, 8.56, -3.8296, -3.68152, 6.9696] & -1 & 2 \\ [5.46210, 5.78, -6.18459, -1.84076, 6.9696] & -0.5 & 2 \\ [6.9696, 0, -6.9696, 0, 6.9696] & 0 & ? \\ [5.46210, -5.78, -6.18459, 1.84076, 6.9696] & 0.5 & 2 \\ [1.6896, -8.56, -3.8296, 3.68152, 6.9696] & 1 & 2 \\ [-2.09789, -5.34, 0.0954, 5.52229, 6.9696] & 1.5 & 1 \\ [-2.1504, 6.88, 5.5904, 7.36305, 6.9696] & 2 & 1 \\ [6.78209, 31.1, 12.6553, 9.20382, 6.9696] & 2.5 & 0 \\ [31.4496, 70.32, 21.2904, 11.0445, 6.9696] & 3 & 0 \end{array}\right]$$

We note that this polynomial has 4 real zeros, one zero in each of the intervals $(-2.5, 2)$, $(-1.5, -1)$, $(1, 1.5)$, and $(2, 2.5)$. Also the question mark occurs where $x = 0$ since a value in the sequence for that value is zero and *Derive* returns SIGN(0) as ±1 which in turn confuses the sum.

Example 2.

In this example we will consider a polynomial with two distinct zeros which are quite close to each other.
Which will give the Sturm sequence and must be defined as $S(x)$ before the search statement may be executed.

A portion of the table giving the results of the search is now presented

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SEARCH(x, 4, 1.25, 1.35, 0.1)
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$$\left[\begin{array}{ccc} \left[0.0121871, -0.445376, -0.0230437, 0.471505, -9.76874\cdot 10^{-5}\right] & 1.25 & 3 \\ \left[0.00824350, 0.372411, 0.0101425, -0.385979, -9.76874\cdot 10^{-5}\right] & 1.35 & 1 \end{array}\right]$$

Which shows that there 2 sign changes between $x = 1.25$ and $x = 1.35$
and tells us that there are two distinct zeros between these two values of x. Newton's method yields $x \approx 1.30008$ and $x \approx 1.31008$ for the zeros of $p(x)$.

Example 3.

This example will show a case where $p(x)$ has a zero of multiplicity 2.

$$P(x) := x^3 - 4.2 \cdot x^2 + 5.61 \cdot x - 2.42$$

STURM(P(x), x, 3)

$$\left[x^3 - 4.2 \cdot x^2 + 5.61 \cdot x - 2.42,\ 3 \cdot x^2 - 8.4 \cdot x + 5.61,\ 0.018 \cdot (10 \cdot x - 11),\ 0\right]$$

The last *Derive* statement indicates that $10x-11$ is the greatest common divisor of $p(x)$ and $p'(x)$. Thus $\frac{p(x)}{10x-11}$ has only simple zeros which are the same as the zeros of $p(x)$.